

# Phonons in Polymeric Quantum Mechanics\*

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Crystal is an important object of study in polymeric quantum mechanics. Based on this and with the goal to study spacetime in little scales with a representation using in loop quantum gravity. The structure of a crystal is used in polymeric quantum mechanics [1], and we use this idea for understanding the spacetime structure using these kinds of representation [2] that change the usual quantum mechanics structure. Micro-scales can represent a problem in general because these type of scales are difficult to manage [3], but Planck scale improves the understanding of polymeric quantum mechanics. It is using that idea; we consider a crystal structure to describe Planck order spacetime in a crystal structure. Classical mechanics describe solid structure for a crystal, but in loop quantum gravity, we establish a description of spacetime. We focus in to study path integral that is used in order to build quantum mechanics within polymeric quantum mechanics formalism. Path integral and polymeric quantum mechanics are interesting themes because one establishes a quantum formulation and other studies a quantum mechanics that establish a different formulation. This idea was to establish as an option initially in construction of quantum mechanics that is used in the formulation of polymeric quantum mechanics that is briefly revised in this work for construct path integral and to analyze this type of formulation in this frame.

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## 86 I. INTRODUCTION

87 Crystal frame is a structure that is adopted in polymeric quantum mechanics and loop quantum mechanics to  
88 describe spacetime [4]. It is common for universe to get grouped with complex form [5] that has been described in  
89 mathematics and physicist. Always it has been used a complex transformation in mathematics in order to establish  
90 behaviours of natural structures [5, 7–9]. As an example of this fact is structures of carbon, fullerene solid, that can  
91 have a preferential direction [10]. Each crystal has a peculiar structure that is interesting to analyze, if we want to  
92 understand collective properties. The general structures to macroscopic scale produce collective phenomena that are  
93 measurable on a big scale, but in little scale also [5, 11]. In mathematics exist a description that describes any scale  
94 that is tried with complex numbers and fractals [12].

95 In general, it is important to make a description of a crystal with a reduction that allows to establish the principal  
96 character of the system. It is at this point where we consider a simple crystal and to make a description with  
97 mechanical qualities. The spacetime has itself properties that can influence in properties of movement for a body  
98 or particle. The fractal is considered because the relationship between different scales and how mass orders that  
99 is similar for certain scales, the universe seems a fractal [12, 13]. In order to establish a structure that is near to  
100 physics independent of scale orders, in this work, we present a non-Taylor series for polymeric quantum mechanics,  
101 Euler series of trigonometric functions [14], and we building a structure fractal that is given by polymeric mechanics  
102  $p \rightarrow \frac{1}{\mu_0} \sin(\frac{\mu_0}{\hbar} p)$  that forms a fractal in nature. In order to use this approximation, we must use the approximation  
103 [6, 15–17] and we use polymeric quantum mechanics as in loop quantum mechanics is used [18, 19] and for special  
104 cases [22–26]. In section II we establish the main problem that is analyzed in this work. We consider a crystal that  
105 is studied and explored more frequently in common conditions. This analysis no is strange for physics of materials  
106 and is recorded in literature, but is instructive to consider. Also, we present dynamics and mechanics in polymeric  
107 quantum mechanics for a linear harmonic oscillator with eigenfrequencies that are naturals in this system. We find  
108 natural frequencies and obtain disengage Hamiltonian with quantum expression with different frequencies resulting  
109 in a linear crystal.

110 we introduce in section II classical formulation for a system of resorts coupled that is main point. In section III we  
111 work with low energies, and it is reviewed a system of particles arranged in a straight line. Particles as collective are  
112 studied looking for new properties in polymer mechanics. In section VIII it is establish dimension for a fractal with  
113 complex structure for classical mechanics. We consider Julia method, that is using complex functions with constraints.  
114 It is established as consider Klein-Gordon model and it is recapitulated models that are basic to quantum mechanics  
115 views as higher order models that are equivalents to usual classical mechanics, and useful for studied universe models.

116 For this case, it is consider a fractal ordering, and as the nature is organized [12]. In section III C, we obtain disengage  
117 Hamiltonian We observe that this Hamiltonian [27] is associated with higher-order where we develop polymeric  
118 mechanics. In section X, it is analyzed a structure in a particle in a box using higher order theories nothing that is  
119 equivalent to usual form in quantum mechanics.

We obtain an integral that is reduced to chaos; if it doesn't limit of integration aren't well defined. In section XII we present conclusions and in section XIII it is consider fractal mathematics.

In the last section, we establish and develop path integral in polymeric quantum mechanics and we study this case.

## II. DISCRETE ROPE

It is instructive to begin mechanics of a chain of oscillators to analyze the main idea for a collective phenomenon that is represented by a collective that is separated in a fundamental oscillator. The general idea is to find a collective phenomenon.

In this discussion, we consider  $N$  coupled oscillators between walls. This system represents a crystal. Lagrangian for this situation is

$$L = \sum_{i=1}^{n+1} \frac{m}{2} \dot{x}_i^2 - \frac{k}{2} (x_i - x_{i-1})^2, \quad (1)$$

where initial conditions are  $x_0 = 0$  and  $x_{n+1} = 0$ , but walls are fixed.

Movement equations are obtained in the habitual way

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0, \quad (2)$$

resulting

$$m\ddot{x}_j - k(x_{j-1} - 2x_j + x_{j+1}) = 0, \quad (3)$$

where  $j$  from 1 to  $n$ .

Next section is solving the last problem, but it is used normal modes. Classical mechanical solutions are

$$x_j = a_j \exp i\omega t, \quad (4)$$

and the secular equation is

$$-ka_{j-1} + (2k - m\omega^2)a_j - ka_{j+1} = 0, \quad (5)$$

where  $a_j$  is a complex number that is used to find solutions  $x_j$ . Then we obtain a pair of solutions from complex structure. We can obtain

$$a_{jr} = a_r \cos(j\gamma_r - \delta), \quad (6)$$

and from (5) we get

$$a_0 = a_{n+1} = 0, \quad (7)$$

to find the system frequencies with the next form

$$\omega_r = 2\sqrt{\frac{k}{m}} \sin\left(\frac{r\pi}{2(n+1)}\right), \quad (8)$$

$$x_j(t) = \sum_{r=1}^n \beta_r a_r \sin\left(j\frac{r\pi}{(n+1)}\right) \exp i(\omega_r t). \quad (9)$$

The last equation has two real equations getting  $x_j(t)$ . An important point is to consider coupled solutions in terms of decoupled form

$$x_j(t) = \sum_{r=1}^n \beta_r a_r \sin\left(j\frac{r\pi}{(n+1)}\right) y_r, \quad (10)$$

$$y_r = \exp i(\omega_r t). \quad (11)$$

Lagrangian (1) in a normal form is

$$L_{MN} = \sum_{r=1}^n \frac{m}{2} \dot{y}_r^2 - \frac{k_r}{2} y_r^2, \quad (12)$$

with Hamiltonian

$$H_{MN} = \sum_{r=1}^n \frac{1}{2m} p_r^2 + \frac{m\omega_r^2}{2} y_r^2, \quad (13)$$

where  $r = 1, \dots, n$ ,  $k_r = m\omega_r^2$ ,  $p_r = m\dot{x}_r$ . The expression  $y_r$  is obtained from (9).

With this idea in mind, we construct a fractal, and propose a generalization to complex functions that permits this structure.

### III. — ENERGIES IN A INFINITY STRING AND MECHANICAL QUANTUM REVIEW

It is important to establish a limit that it takes to Quantum field formulation for this chain or string of oscillators and to study briefly in order to establish it in polymeric quantum mechanics.

Low energies have little displacements  $|x_i - x_{i-1}| \rightarrow a$  where  $a = \Delta l$  is an interval. Using the above the continuum limit can be obtained. From equation (1) and with little displacement approximations

$$L = \sum_{i=1}^{n+1} \left[ \frac{m}{2} \dot{\varphi}_i^2(x) - \frac{k}{2} (\varphi_i(x) - \varphi_{i-1}(x))^2 \right] \rightarrow \frac{1}{l} \int dx \left[ \frac{\rho}{2} \dot{\varphi}^2(x) - \frac{ka^2}{2} (\partial_x \varphi(x))^2 \right], \quad (14)$$

and given a continuum medium as this string formed by harmonic oscillators the Lagrangian density is

$$\mathcal{L} \approx \left[ \frac{\rho}{2} \dot{\varphi}^2(x) - \frac{ka^2}{2} (\partial_x \varphi(x))^2 \right], \quad (15)$$

with  $\omega_n = n\omega_1 = n\omega$  that in the Hamiltonian formulation is obtained a Hamiltonian

$$H = \frac{1}{l} \int_0^l dx \left[ \frac{1}{2\rho} \pi^2(x) + \frac{n^2\omega^2}{2} (\partial_x \varphi(x))^2 \right], \quad (16)$$

where in a continuum medium and  $\omega = \frac{k}{\rho}$  we have a density

$$\mathcal{H} = \left[ \frac{1}{2\rho} \pi^2(x) + \frac{ka^2}{2} (\partial_x \varphi(x))^2 \right], \quad (17)$$

and for this string or chain of oscillators  $l = an$  and momentum

$$\pi = \rho \dot{\varphi}, \quad (18)$$

and movement equations for this string or chain of oscillators are

$$\begin{aligned} \dot{\pi}(x) &= \frac{\partial \mathcal{H}}{\partial \varphi(x)} = ka^2 \frac{\partial^2}{\partial x^2} \varphi(x), \\ \dot{\varphi}(x) &= \frac{\partial \mathcal{H}}{\partial \pi(x)} = \frac{1}{\rho} \pi(x), \end{aligned} \quad (19)$$

that in Lagrangian formalism has movement equations and

$$\rho \ddot{\varphi}(x) - ka^2 \frac{\partial^2}{\partial x^2} \varphi(x) = 0, \quad (20)$$

is wave equation and we can define  $c_g^2 = \frac{T}{\rho}$  and  $T = ka^2$  that is square velocity

$$\ddot{\varphi}(x) - c_g^2 \frac{\partial^2}{\partial x^2} \varphi(x) = 0, \quad (21)$$

here it is an option to use creation and annihilation operators. It is important to mention  $c_g$  is group velocity in this structure that as is used in polymeric mechanics is light velocity. We appeal in quantum structure for this work and establish creation and annihilation operator, used in a Fourier transformation. These operators are supported for this construction.

### A. Classical Crystal Structure in a Dimension

In this subsection, we establish a Hamiltonian density for wave functions; this density is a mix between waves and opposite waves. This density has a mix due model structure that is a line of oscillators with rigid boundaries. In order to favour and to clarify the nature of phonons that appears in a structured crystalline that is modelling with a chain of oscillators.

It is necessary to mention that given waves in this model, we proceed to pay attention no in linear harmonic oscillators, that can be decoupled, and in terms of waves also can be decoupled.

Using the Fourier transformation to Hamiltonian by means of the wave function and momentum

$$H = \sum_{\kappa} \left[ \frac{1}{2\rho} \pi_{\kappa} \pi_{-\kappa} - \frac{1}{2\rho} (\partial_x \varphi_{\kappa}) (\partial_x \varphi_{-\kappa}) + \frac{\rho \omega_{\kappa}^2}{2} \varphi_{\kappa} \varphi_{-\kappa} \right], \quad (22)$$

and  $\kappa$  is discrete with linear partition, that results of a problem and boundary structures [32]. Here we suppose a Fourier transformation in one dimension, but there is another dimension where we do not establish a Fourier transformation.

With respect to wavenumber is

$$k_r = \left( \frac{r\pi}{2(n+1)l_P} \right), \quad \text{con } r = 1, \dots, N+1. \quad (23)$$

that permit analyzes properties as energy in this toy model. This system is equivalent to closed chain if we join last nodes forced for the final of this chain. It is important to mention that this Hamiltonian is not diagonal and could have properties due structure that we have reduced to this chain.

### B. Diagonal Hamiltonian Density

To analyze the properties of this system, we separate each type of waves that include a direction.

In order to analyze the structure of a crystal is convenient to use a diagonal expression (22) or in order to quantify we establish a diagonal transformation that is equivalent to the initial model

$$H = \sum_{r=1}^{N+1} \left[ \frac{1}{4\rho} \begin{pmatrix} \pi_{\kappa_r} & \pi_{-\kappa_r} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi_{\kappa_r} \\ \pi_{-\kappa_r} \end{pmatrix} + \frac{\rho \omega_r^2}{4} \begin{pmatrix} \varphi_{\kappa_r} & \varphi_{-\kappa_r} \end{pmatrix} \begin{pmatrix} 0 & 1 + \partial_x \\ 1 + \partial_x & 0 \end{pmatrix} \begin{pmatrix} \varphi_{\kappa_r} \\ \varphi_{-\kappa_r} \end{pmatrix} \right]. \quad (24)$$

where in a natural form appear a matrix that describes this model

$$\mathcal{A} - \lambda 1 = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \quad (25)$$

with determinant  $\det(\mathcal{A} - \lambda 1) = \lambda^2 - 1 = 0$  resulting  $\lambda = +/ - 1$ .

Eigenvalue in this work is used to analyze this system reducing to a system of harmonic oscillators.

The eigenvalues are

$$\begin{aligned} \mathcal{V}_1 &= \sigma \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{Para } \lambda = 1 \\ \mathcal{V}_2 &= \sigma \begin{pmatrix} 1 \\ -1 \end{pmatrix}, & \text{Para } \lambda = -1 \end{aligned} \quad (26)$$

and with eigenvalues, we assemble a matrix

$$\mathcal{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{V}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (27)$$

then

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (28)$$

In another hand for fields, we get

$$\begin{pmatrix} (1 + \partial_x) & 0 \\ 0 & -(1 + \partial_x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & (1 + \partial_x) \\ (1 + \partial_x) & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (29)$$

that is used to form a transformation in this system and establish transformation in Hamiltonian that is no-canonical.

### C. onian

In the last section, we do especial emphasis in a matrix form of eigenvalue for making a non-canonical transformation that is valid in mathematics, but it could have differences with classical mechanics that are used in this work.

Eigenvalues method is used in mathematics and physics, in another frame, to establish a transformation non-canonical [33].

In order to obtain a new base we choose

$$\begin{pmatrix} \varphi_{\kappa_r} \\ \varphi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_{\kappa_r} \\ \psi_{-\kappa_r} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{\kappa_r} & \varphi_{-\kappa_r} \end{pmatrix} = \begin{pmatrix} \psi_{\kappa_r} & \psi_{-\kappa_r} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (30)$$

$$\begin{pmatrix} \pi_{\kappa_r} \\ \pi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pi_{\psi_{\kappa_r}} \\ \pi_{\psi_{-\kappa_r}} \end{pmatrix},$$

$$\begin{pmatrix} \pi_{\kappa_r} & \pi_{-\kappa_r} \end{pmatrix} = \begin{pmatrix} \pi_{\psi_{\kappa_r}} & \pi_{\psi_{-\kappa_r}} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (31)$$

In terms of this basis, it is possible to establish a Hamiltonian that permits to make a description of this linear model of springs join in a linear red. Using this base, we obtain

$$H = \sum_{r=1}^n \left[ \frac{1}{4\rho} \begin{pmatrix} \pi_{\psi_{\kappa_r}} & \pi_{\psi_{-\kappa_r}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \pi_{\psi_{\kappa_r}} \\ \pi_{\psi_{-\kappa_r}} \end{pmatrix} + \frac{\rho\omega_r^2}{4} \begin{pmatrix} \psi_{\kappa_r} & \psi_{-\kappa_r} \end{pmatrix} \begin{pmatrix} (1 + \partial_x) & 0 \\ 0 & -(1 + \partial_x) \end{pmatrix} \begin{pmatrix} \psi_{\kappa_r} \\ \psi_{-\kappa_r} \end{pmatrix} \right]. \quad (32)$$

and we realize a linear transformation where we get

$$\begin{pmatrix} \varphi_{\kappa_r} \\ \varphi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\psi_{\kappa_r} + \psi_{-\kappa_r}) \\ (\psi_{\kappa_r} - \psi_{-\kappa_r}) \end{pmatrix},$$

$$\begin{pmatrix} \varphi_{\kappa_r} & \varphi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\psi_{\kappa_r} + \psi_{-\kappa_r}) & (\psi_{\kappa_r} - \psi_{-\kappa_r}) \end{pmatrix}, \quad (33)$$

$$\begin{pmatrix} \pi_{\kappa_r} \\ \pi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\pi_{\psi_{\kappa_r}} + \pi_{\psi_{-\kappa_r}}) \\ (\pi_{\psi_{\kappa_r}} - \pi_{\psi_{-\kappa_r}}) \end{pmatrix},$$

$$\begin{pmatrix} \pi_{\kappa_r} & \pi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\pi_{\psi_{\kappa_r}} + \pi_{\psi_{-\kappa_r}}) & (\pi_{\psi_{\kappa_r}} - \pi_{\psi_{-\kappa_r}}) \end{pmatrix}, \quad (34)$$

and inverse transformation

$$\begin{pmatrix} \psi_{\kappa_r} \\ \psi_{-\kappa_r} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \varphi_{\kappa_r} \\ \varphi_{-\kappa_r} \end{pmatrix},$$

$$\begin{pmatrix} \pi_{\psi_{\kappa_r}} \\ \pi_{\psi_{-\kappa_r}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pi_{\kappa_r} \\ \pi_{-\kappa_r} \end{pmatrix}, \quad (35)$$

it is important to repeat that we want to establish a non-canonical transformation, and it is considered eigenvalues, a tool too important in mathematics.

To apply this transformation in this Hamiltonian, we take next transformations in non-canonical variables

$$\psi_{\kappa_r} = \frac{1}{\sqrt{2}}(\varphi_{\kappa_r} + \varphi_{-\kappa_r}), \quad \psi_{-\kappa_r} = \frac{1}{\sqrt{2}}(\varphi_{\kappa_r} - \varphi_{-\kappa_r}),$$

$$\pi_{\psi_{\kappa_r}} = \frac{1}{\sqrt{2}}(\pi_{\kappa_r} + \pi_{-\kappa_r}), \quad \pi_{\psi_{-\kappa_r}} = \frac{1}{\sqrt{2}}(\pi_{\kappa_r} - \pi_{-\kappa_r}), \quad (36)$$

where a final expression is

$$H = \sum_{r=1}^n \left\{ \frac{1}{4\rho} \pi_{\psi_{\kappa_r}}^2 - (\partial_x \psi_{\kappa_r})^2 + \frac{\rho \omega_r^2}{4} \psi_{\kappa_r}^2 - \left[ \frac{1}{4\rho} \pi_{\psi_{-\kappa_r}}^2 - (\partial_x \psi_{-\kappa_r})^2 + \frac{\rho \omega_r^2}{4} \psi_{-\kappa_r}^2 \right] \right\}, \quad (37)$$

and we obtain normal modes with negative energy.

It is interesting to mention that this eigenvalue transformation has a relationship a complex canonical transformation that solve the problem of negative energy and this problem has a relationship with higher-order theories before studies in other works [15, 17–20].

#### D. Crystal and Relativistic Wavenumber

it is instructive to consider  $\kappa_r$  as angular wave-number. There is a relationship as in (??), and it is necessary to establish

$$\begin{aligned} \kappa_r &= \frac{r\pi}{2(n+1)l_P}, \quad \text{con } r = 1, \dots, N+1. \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin(k_r l_P) \approx c_s k_r \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ \omega_r^2 &= \left(\frac{4k}{m}\right) \sin^2\left(\frac{1}{2}\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ &= 1 - \cos\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right) \end{aligned} \quad (38)$$

with  $c_s = c = 2\sqrt{\frac{k}{m_P}} l_P$ , and  $a = l_P$  is distance of equilibrium points. This above expression is the equivalent to polymeric quantum mechanics with  $c_s$  as speed of light. It is necessary to mention that is possible to obtain

$$k = k_P = m_P \left(\frac{c}{2l_P}\right)^2, \quad (39)$$

that is as very small time-space distortion.

#### E. Pais-Uhlenbeck and Bernard-Duncan Model

The structure of this simplified crystal has a relationship with highest order formalism that was studied by Pais-Uhlenbeck [27].

To get started, we consider a density Hamiltonian with opposite signs that we viewed is formed by linear oscillators, representing a crystal

$$H = \sum_{r=1}^n \left\{ \frac{1}{4\rho} \pi_{\Psi_r}^2 - (\partial_x \Psi_r)^2 + \frac{\rho \omega_r^2}{4} \Psi_r^2 - \left[ \frac{1}{4\rho} \pi_{\Psi_{-r}}^2 - (\partial_x \Psi_{-r})^2 + \frac{\rho \omega_r^2}{4} \Psi_{-r}^2 \right] \right\}, \quad (40)$$

and It was studied in [27] where it is proposed a non-local transformation

$$\begin{aligned} \psi_r &= \frac{(\square + \rho \omega_r^2)}{2(2\epsilon \rho \omega_r)^{\frac{1}{2}}} \varphi_r, & \psi_{-r} &= \frac{(\square + (1+2\epsilon)\rho \omega_r^2)}{2(2\epsilon \rho \omega_r)^{\frac{1}{2}}} \varphi_r, \\ \pi_{\Psi_r} &= \frac{(\square + \rho \omega_r^2)}{2((1+2\epsilon)\rho \omega_r)^{\frac{1}{2}}} \dot{\varphi}_r, & \pi_{\Psi_{-r}} &= \frac{(\square + (1+2\epsilon)\rho \omega_r^2)}{2((1+2\epsilon)\rho \omega_r)^{\frac{1}{2}}} \dot{\varphi}_r, \end{aligned} \quad (41)$$

with  $\omega_{2r}^2 = \omega_r^2 + 2\epsilon \omega_r$ . It is important to mention that is included momenta in these transformations.

We note that last model (118) has a non-local relationship with [27], Pais Uhlenbeck model that is named as Bernard Duncan model for fields

$$\mathcal{L}_{BD} = \sum_{r=1}^n \left\{ -\frac{1}{2} (\Box \varphi_r)^2 + (1 + \epsilon) \rho \omega_r \partial_\mu \varphi_r \partial^\mu \varphi_r - \frac{(1 + 2\epsilon) \rho^2 \omega_r^4}{2} \varphi_r^2 \right\}, \quad (42)$$

that is Pais-Uhlenbeck model in particles and Bernard-Duncan model for fields.

where we observe that the higher order model has relationship with Klain-Gordon Hamiltonian with opposite sign that produces negative norm states. In this point, we must consider to establish a different form that describes a crystal, in our case linear oscillators, thinking space-time as a crystalline structure [20].

## F. No Canonical Transformation

There is a transformation where it can decouple particles of a crystal by means eigenvalues [33] that in this case is obtained Klein Gordon fields oscillating with different frequencies.

The new disengage system has new wave functions

$$\begin{aligned} \varphi_j(x) &= \sum_{r=1}^n \beta_r a_r \sin\left(j \frac{r\pi}{(n+1)}\right) \Psi_r, \\ \varphi_j^\dagger(x) &= \sum_{r=1}^n \beta_r^\dagger a_r^\dagger \sin\left(j \frac{r\pi}{(n+1)}\right) \Psi_r^\dagger, \\ \pi_{\varphi j} &= \sum_{r=1}^n \beta_r a_r \sin\left(j \frac{r\pi}{(n+1)}\right) \Pi_r, \\ \pi_{\varphi j}^\dagger &= \sum_{r=1}^n \beta_r^\dagger a_r^\dagger \sin\left(j \frac{r\pi}{(n+1)}\right) \Pi_r^\dagger, \end{aligned} \quad (43)$$

or in a Hamiltonian Transformation

$$H_{MN} = \sum_{r=-n}^n \frac{1}{2\rho} \Pi_r^2 + \frac{\rho \omega_r^2}{2} \Psi_r^2, \quad (44)$$

with  $k_r = \rho \omega_r^2$ ,  $\Pi_r = \rho \dot{\Psi}_r$  and field  $\Psi_r$  is (121). It is observed a relationship with the harmonic oscillator. These coordinates are not a canonical system, but it is a way of normal modes that we can use to describe in spacetime with polymeric quantum mechanics. It is necessary to mention that with this change we have a positive sign in Hamiltonian, and there is a no-canonical transformation. However, it exists a complex canonical transformation [6, 15, 16].

## G. Energy Deformation and Space-Time.

It is important to include [?] with Hamiltonian form in path integral. The Lagrangian form is useful, but in our case not permit to note the structure. Then we use (116) and obtain

$$E_{n,r}^2 = \hbar^2 \omega_{n,r}^2 = \Pi_{n,r}^4 c^2 + m^2 c^4, \quad (45)$$

that is the relativistic energy

$$\omega_{n,r}^2 = \sin^2\left(\frac{l_P}{c} \omega_{n,r}\right) = \frac{4k}{m_P} \left(1 - \cos\left(\frac{m_P}{k} c \kappa_r\right)\right) \quad (46)$$

$$\approx \frac{c^2}{\hbar^2 \mu_0^2} \left(1 - \cos\left(\frac{\mu_0}{\hbar} \Pi_r\right)\right)^2 + \frac{m_P^2 c^4}{\hbar^2}, \quad (47)$$

that is the oscillation frequency for space-time, and it is possible to calculate the spring constant for the space-time

$$k \approx \frac{m_P c^2}{4\hbar^2 \mu_0^2} = \sqrt{\frac{\hbar c^5}{16G\mu_0^4}}, \quad (48)$$

that is a deformation in loop quantum mechanics.

## H. Planck Length

We analyze small scales where it is difficult to establish a measure if it crosses scales of quantum mechanics, and there are conceptual problems and difficult managing.

In order to establish this description, we don't take a limit to  $\mu_0 \rightarrow 0$  instead we take the smallest quantities [3, 4, 21]

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \quad m_P = \sqrt{\frac{\hbar c}{G}} \quad (49)$$

and with wave numbers that is established for this linear crystal We use wave number and it is established a new quantity that define scales and is used in polymeric mechanics (133) as

$$\mu_0 = \frac{\pi}{2(n+1)l_P} = \frac{\pi c^3}{2(n+1)\hbar G}, \quad (50)$$

that in this case is used for a red of little oscillations that represent spacetime structure.

## IV. FRACTAL DIMENSION

It is important to mention as is defined a dimension in a fractal using

$$D = \lim_{\epsilon \rightarrow 0} \frac{\log |S(\epsilon)|}{\log |\frac{1}{\epsilon}|}, \quad (51)$$

that is Hausdorff- Besicovitch dimension and  $S(\epsilon)$  is number of self-similar structures with linear side  $\epsilon$  that is necessary to cover structure.

With this idea is possible to construct a fractal that establish chaos. If we consider

$$\pi_r = \frac{1}{\mu_0} \sin(\mu_0 \pi_r), \quad (52)$$

where is supposed that momentum is complex in general []. Fractal dimension as is proposed by Hausdorff- Besicovitch is .

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that in this case is used for a red of little oscillations that represent space-time structure.

## V. FRACTAL STRUCTURE WITH JULIA METHOD

If it exists a complex structure of canonical transformation in canonical structure [6, 15, 17, 18, 20], we could think in a fractal structure for nature. We think it exist a ordering of nature that forms a fractal structure that is noted in each scale. No confuse particles that there are with structures in universe. It is a way of ordering, no physical object. Thinking in ordering for nature, including space, waste less energy repeat structures by means of fractals, but it is not a fundamental structure because ordering is not physical object. Space-time complies with this rule of minimal ordering that is perceptive where in this ordering matter is more establish.

The Julia set are fractals that are obtained studying iteration of holomorphic function for complex numbers that could present in complex canonical mechanics [15–18] and quantum structure where complex numbers prevalent.

It is possible to define a set called Julia set  $\mathcal{J}$  and it is an holograph could be written as the Julia set that depends of function  $f$  and  $\mathcal{J}(f)$ . The general idea is to begin the complex iteration

$$z = f(z),$$

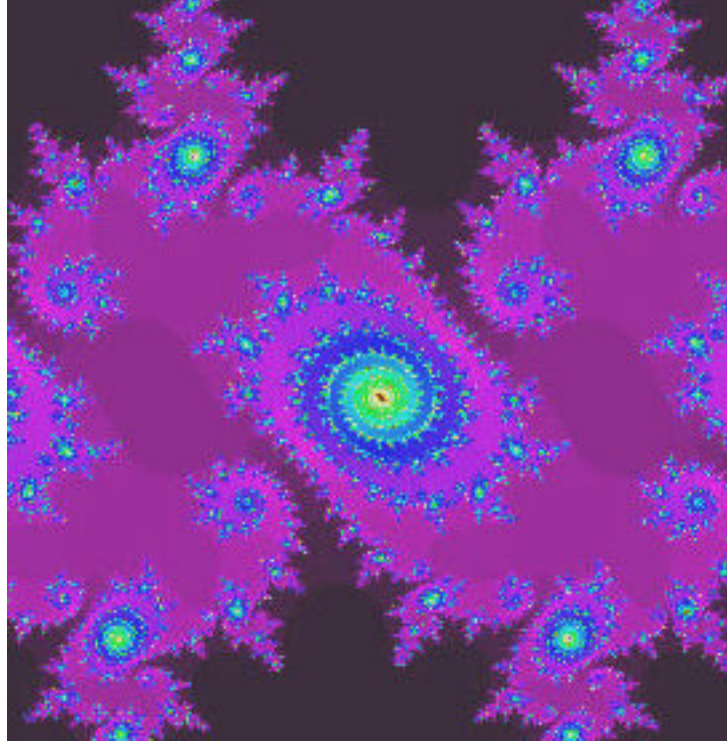


FIG. 1. Julia set of fractal associated to this frequency. The images is from <https://oocities.org/valcocy/sinz.html>

in this way for example iterating 3 times

$$z = f(f(f(z))),$$

that in terms of  $\sin(x)$  or  $\cos(x)$  and Euler's products [14] could generate an alternative way to Taylor approximate. An important fact is that if  $\|z_n\| > 2$ , then succession is divergent and  $\|z\|$  does not belong to Julia set [12, 13]. Other way of writting this succession is

$$z_{n+1} = f(z_n),$$

In terms of frequencies we think that we have an extended complex quantities, no real quantities as Bender works.

### A. Julia's Fractal and Phonons

In order to describe a complicated structure as a phonon, that is a virtual particle that help to describe a red of particles or properties of space-time that can be described with Planck mass and Planck length with frequencies in a structure, frequency in this case depend of structure.

It is necessary to consider frequency as a structure fractal

$$\omega_r = 2\sqrt{\frac{k}{m}} \sin\left(\frac{a}{c}\omega_r\right), \quad (54)$$

here we include structure of fractal that can result contradictory, but It is supposed that is in different scales correct. In summary, we use this notation in order to construct a fractal Fig. 2. Fractal that we generate is Julia's fractal is more common in lowest velocities.

It is instructive to consider a mechanical model of this chain of oscillators. We consider differences between the closest wave numbers in this model

$$\begin{aligned} \mu_0^{cl} = \kappa_r - \kappa_{r-1} &= \left( \frac{r\pi}{2(n+1)a} - \frac{(r-1)\pi}{2(n+1)a} \right) \\ &= \frac{\pi}{2(n+1)a}, = \frac{\pi}{2l_P}, \end{aligned} \quad (55)$$

that is important to improve this description and realize structures in next quantum model with polymer quantum mechanics.

### B. Particle In a Box

Considering a particle in a box with a border in  $x = 0, L$  in quantum mechanics. We consider Schrodinger equation for a free particle

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(t, x), \quad (56)$$

and it is usual for solve this equation

$$\begin{aligned} \Psi(t, x) &= \phi(t) \psi(x) \\ i\hbar \frac{\partial}{\partial t} \phi_n(t) &= E_n \phi_n(t), \end{aligned} \quad (57)$$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n = E_n \psi_n, \text{ with } 0 < x < L \quad (58)$$

with solutions

$$\begin{aligned} \phi_n(t) &= \exp\left(-i \frac{E_n}{\hbar} t\right), \\ \psi_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{\sqrt{2mn}\pi x}{\hbar L}\right), \end{aligned} \quad (59)$$

but this equation has two solutions that describe two possible solutions as Schrodinger equation only consider a part. In this case infinity walls is a hypothetical until case because it is illustrative with opposite direction solutions. The extra opposite solution in a box is

$$\psi_{-n}(x) = \sqrt{\frac{2}{L}} \sin\left(-\frac{\sqrt{2mn}\pi x}{\hbar L}\right), \quad (60)$$

and the first equation (135) has to had a part temporal that evolves

$$\frac{d^2}{dt^2} \phi_n(t) + \frac{c^2 E_n}{\hbar} \phi_n(t) = 0, \quad (61)$$

that in case of this equation, we must consider solution between this two walls for Schrodinger equation where is known

$$[\partial_t - c_s \partial_x - \frac{E_n}{\hbar}][\partial_t + c_s \partial_x + \frac{E_n}{\hbar}] \Psi(t, x) = 0 \quad (62)$$

we agree that is a linear combination from the first case, but in order to consider another complete set solution in this set is

$$\Psi_n = \psi_n + \psi_{-n} = \sqrt{\frac{2}{L}} [\sin\left(\frac{n\pi}{L}(x - c_s t)\right) + \sin\left(-\frac{n\pi}{L}(x + c_s t)\right)] \Theta(t_w - t) \quad (63)$$

that is a solution for a higher order theory []. Total solution is an algebraic solution

$$\Psi_n = \sum_{j=1}^n \psi_j = \sqrt{\frac{2\pi}{L}} \sum_{j=1}^n [\sin\left(\frac{j\pi}{L}(x - c_s t)\right) + \sin\left(-\frac{j\pi}{L}(x + c_s t)\right)] \quad (64)$$

$$= \sqrt{\frac{2}{L}} \left[ \frac{2}{\sin\left(\frac{(x - c_s t)}{2}\right)} \sin\left(\frac{(n+1)\pi}{2L}(x - c_s t)\right) \right] \sin\left(\frac{n(x - c_s t)}{2}\right), \quad (65)$$

where we mention second solution is dependent linear of first solution (142) with negative solution for appositve's moving sign that is important to consider, because particles are maintained between infinity walls. Energy for free particles in quantum mechanics is

$$E_r = \frac{p_r^2}{2m} = \frac{\hbar^2 k_r^2}{2m} = \sum_{\kappa=-r}^r E_\kappa = \sum_{\kappa=-r}^r \frac{\hbar^2 \pi^2}{mL^2} n_\kappa^2 = \frac{\hbar^2 \pi^2 (2r+1)}{mL^2} \frac{r(r+1)}{6}, \quad (66)$$

that is a dependent non-linear term including in this base. It is important to mention that we have  $-j$  a return wave that is dependent of wave in the first part. Only walls could absorb energy, but in this point we suppose that this exchange is not possible.

### 1. Klein-Gordon Equation

It is important to consider relativistic energy

$$E^2 = p^2 c^2 + m^2 c^4, \quad (67)$$

where we promote quantum operators to dynamics variables

$$\hat{E} = \hat{p}_0 = i\hbar \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -i\hbar \nabla, \quad (68)$$

$$\frac{\partial^2}{\partial t^2} \Psi(t, x) - c^2 \frac{\partial^2}{\partial x^2} \Psi(t, x) = -\frac{m^2 c^4}{\hbar^2} \Psi(t, x) \quad (69)$$

$$\Psi(t, x) = \phi(t) \psi(x), \quad (70)$$

$$\frac{1}{\phi(t)} \frac{\partial^2}{\partial t^2} \phi(t) = -\left(\frac{E^2}{\hbar^2}\right) \quad (71)$$

$$\frac{1}{\psi(x)} \frac{\partial^2}{\partial x^2} \psi(x) = -\left(\frac{p^2 c^2}{\hbar^2}\right). \quad (72)$$

Solutions for this equations are

$$\phi(t) = e_0 \exp\left(i \frac{E}{\hbar} (t - t_0)\right) + e_1 \exp\left(-i \frac{E}{\hbar} (t - t_0)\right), \quad (73)$$

$$\psi(x) = P_0 \exp\left(i \frac{cp}{\hbar} (x - x_0) + \delta_0\right) + P_1 \exp\left(-i \frac{cp}{\hbar} (x - x_0) + \delta_1\right), \quad (74)$$

In a quantum level Klein-Gordon is not feasible option as it produce negative probabilities, but for quantum field theory it exists.

Another hand this solutions for position is

$$\psi(0) = \psi(L) = 0, \quad (75)$$

$$\psi_+(x) = P_0 \left[ \cos\left(\frac{cp}{\hbar} x + \delta_0\right) + i \sin\left(\frac{cp}{\hbar} x + \delta_0\right) \right], \quad (76)$$

$$\psi(0) = \cos(\delta_0) + i \sin(\delta_0), \quad \psi(L) = \cos\left(\frac{cp}{\hbar} L + \delta_0\right) + i \sin\left(\frac{cp}{\hbar} L + \delta_0\right), \quad (77)$$

$$\psi(L) = \cos\left(\frac{cp}{\hbar} L\right) \cos(\delta_0) - \sin\left(\frac{cp}{\hbar} L\right) \sin(\delta_0) + i \sin\left(\frac{cp}{\hbar} L\right) \cos(\delta_0) + i \sin(\delta_0) \cos\left(\frac{cp}{\hbar} L\right) \quad (78)$$

$$\psi(L) = \cos(\delta_0) \left[ \cos\left(\frac{cp}{\hbar} L\right) + i \sin\left(\frac{cp}{\hbar} L\right) \right] + i \sin(\delta_0) \left[ \cos\left(\frac{cp}{\hbar} L\right) + i \sin\left(\frac{cp}{\hbar} L\right) \right] \quad (79)$$

It is important to mention that in the wall when  $x = 0$  then we obtain that  $\delta_0 = 0$

$$\psi_+(L) = \left[ \cos\left(\frac{cp}{\hbar} L\right) + i \sin\left(\frac{cp}{\hbar} L\right) \right] = \exp\left(i \frac{cp}{\hbar} L\right), \quad \psi_+(0) = 1, \quad (80)$$

and

$$p_j = \frac{\pi \hbar j}{L}, \quad \text{Momentum for particle in a box where it depends of } L \text{ size of box.} \quad (81)$$

$\wp_j = \frac{\pi \hbar j}{L}$ , Momentum for a bounced particle that is bounded in a box. This wave front can be altered by second wall. In this model there is not friction and in walls there are not perfect rigid limits. (82)

and total energy including each n-particle in box is

$$\sum_{j=0}^n E_j = Lc^2 \frac{\hbar^2 \pi^2}{2mL^2} \sum_{j=-n}^n j^2 + (n+1)m^2 c^4 = \frac{\hbar^2 \pi^2}{mL} \frac{n(n+1)(2n+1)}{6} c^2 + (n+1)m^2 c^4, \quad (83)$$

where in case of negative movement, we get

$$\psi_-(L) = \cos(\delta_-) [\cos(\frac{\mathcal{O}}{\hbar}L) - i \sin(\frac{\mathcal{O}}{\hbar}L)] + i \sin(\delta_-) [\cos(\frac{\mathcal{O}}{\hbar}L) - i \sin(\frac{\mathcal{O}}{\hbar}L)] = \exp -i(\frac{\mathcal{O}}{\hbar}L), \quad \psi_-(L) = 0, \quad (84)$$

and it is established phases  $\delta_- = 0$  and  $\delta_- = 2\pi$

$$\psi_+(x) = \cos(\frac{\pi j_+}{L}x) - i \sin(\frac{\pi j_+}{L}x), \quad (85)$$

$$\psi_-(x) = \cos(\frac{\pi j_-}{L}xi) - i \sin(\frac{\pi j_-}{L}x), \quad (86)$$

with wave zero in walls

$$\psi_+(x) = \sin(\frac{\pi j_+}{L}x), \quad \psi_-(x) = -\sin(\frac{\pi j_-}{L}x), \quad (87)$$

that is total solution of Klein-Gordon equation

$$\Psi(x) = \psi_+(\frac{\pi j_+}{L}x) + \psi_-(\frac{\pi j_-}{L}x), \quad (88)$$

that of course both solutions are dependent linear, but we do not abother way to say that both waves go in opposite directions.

### C. Relativistic Dispersion

It is usual in modern physics to associate energy with frequency and wave number with momentum. Here we consider a structure in a little crystal, where space-time in small scales is considered with a representation of quantum mechanics according to description. we consider first a relationship between frequency, wave number and relativistic description

$$\omega^2 = \kappa^2 c^2 + \frac{c^4}{\hbar^2} m^2, \quad (\omega - \kappa c)(\omega + \kappa c) = \frac{c^4}{\hbar^2} m^2, \quad (89)$$

$$(\omega - \kappa c) = \frac{c^2}{\hbar} m, \quad (\omega + \kappa c) = \frac{c^2}{\hbar} m, \quad (90)$$

where dispersion relationship has two directions, that include opposite direction.

### D. Phonons In a Box

In order to handle difficult cases, we consider phonons method that is useful in this situation

$$\mathcal{H} = \sum_{n=-r}^r \mathcal{H}_n, \quad (91)$$

$$\mathcal{H}_n = \frac{1}{2m} \pi_n^2 + \frac{m\omega_n^2}{2} \psi_n^2, \quad (92)$$

$$\omega_n = 2\sqrt{\frac{k}{m}} \sin(\frac{\pi n}{2L}), \quad \omega_n^2 = \frac{4k}{m} (1 - \cos(\frac{\pi n}{2L})), \quad (93)$$

where we use harmonic oscillator frequencies

$$\Psi(x) = \prod_{n=-r}^r \psi_n(x), \quad (94)$$

that is applied in quantum mechanics and it is a way of describing a system for a lot of particles.

### E. Phonon Hamiltonian In a Box

we build a system of virtual particles in an usual way with a infinity line of harmonic oscillators  $n \rightarrow \infty$ .

$$\mathcal{H}_n = \frac{1}{2m}\pi_n^2 + \frac{m\omega_0^2 n^2}{2}\psi_n^2, \quad (95)$$

$$L = (N + 1) \approx N \quad \text{If } N \gg 1 \text{ is very huge.} \quad (96)$$

where frequencies are regulated by quantity of particles

$$\omega_n = \sqrt{\frac{k}{m}} \sin\left(\frac{\pi n}{2L}\right) \approx \sqrt{\frac{k}{m}} \left(\frac{\pi n}{2L}\right) \quad \omega_n^2 = \frac{4k}{m} \left(1 - \cos\left(\frac{\pi n}{2L}\right)\right), \quad (97)$$

$$\lim_{N \rightarrow \infty} \sin\left(\frac{\pi n}{2L}\right) \approx \frac{\pi n}{2L} \quad \omega_0 = \frac{\pi}{2L} \quad (98)$$

with  $L = N + 1$  and  $\omega_1 = \omega_0, \omega_2 = 2\omega_0, \omega_3 = 3\omega_0, \dots$ . We can note

$$\mathcal{H}_n = \left(\frac{1}{2m}\pi_n^2 + \frac{m(c^2\kappa^2 + \frac{m^2c^4}{\hbar^2})}{2}\psi_n^2\right), \quad (99)$$

and wave functions for this system

$$\psi(x) = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi\sqrt{2m}}{\hbar L}x\right) + \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \sin\left(-\frac{\pi\sqrt{2m}}{\hbar L}x\right), \quad (100)$$

$$p = \frac{\pi\sqrt{2m}}{\hbar L}, \quad \wp = -\frac{\pi\sqrt{2m}}{\hbar L}. \quad (101)$$

and as frequencies we handle momentum, this can be most natural

$$p_r = \frac{2\hbar}{c} \sqrt{\frac{k}{m}} \sin\left(\frac{r\pi}{2(n+1)a}\right), \quad (102)$$

but exist an alternative way with higher order theories that describe wave function that bounces in a wall.

### VI. PARTICLE IN A BOX WITH HIGHER ORDER THEORIES

In order to establish a system that includes wave function that bounces, we studies Klein-Gordon equation for free particle in a box

$$\hat{E}\phi(t) = (c\kappa + \frac{mc^2}{\hbar})\phi(t), \quad \hat{E}\phi(t) = (\frac{mc^2}{\hbar} - c\kappa)\phi(t), \quad (103)$$

$$(i\frac{\partial}{\partial t} - c\tilde{\kappa})\phi_1(t) = \frac{c^2}{\hbar}m\phi_1(t), \quad (i\frac{\partial}{\partial t} + c\kappa)\phi_2(t) = \frac{c^2}{\hbar}m\phi_2(t), \quad (104)$$

$$\tilde{\kappa} = \kappa_{-,j} = \frac{\pi j_-}{L}, \quad \kappa_{+,j} = \frac{\pi j_+}{L}, \quad (105)$$

with wave functions

$$\phi_1(t) = \alpha_1 \exp\left(-\frac{it(mc^2 + c\hbar\tilde{\kappa})}{\hbar}\right), \quad \phi_2(t) = \alpha_2 \exp\left(-\frac{it(mc^2 - c\hbar\kappa)}{\hbar}\right), \quad (106)$$

that no-necessary are unitaries.

$$(E - \frac{c\hbar}{i} \frac{\partial}{\partial x})\psi_1(x) = c^2 m \psi_1(x), \quad (E + \frac{c\hbar}{i} \frac{\partial}{\partial x})\psi_2(x) = c^2 m \psi_2(x), \quad (107)$$

$$\psi_1(x) = k_1 \cos\left(\left[\frac{E_- - mc^2}{\hbar c}\right]x\right) + ik_1 \sin\left(\left[\frac{E_- - mc^2}{\hbar c}\right]x\right), \quad (107)$$

$$\psi_2(x) = k_2 \cos\left(\left[\frac{E_+ - mc^2}{\hbar c}\right]x\right) - ik_2 \sin\left(\left[\frac{E_+ - mc^2}{\hbar c}\right]x\right), \quad (108)$$

with frequencies

$$\begin{aligned}\omega_- &= mc^2 + \hbar c \kappa_-, & \omega_+ &= mc^2 - \hbar c \tilde{\kappa}_+, \\ \omega_- &= \frac{mc^2}{\hbar} + c \kappa, & \omega_+ &= \frac{mc^2}{\hbar} - c \tilde{\kappa},\end{aligned}\quad (109)$$

with bounding conditons

$$\psi_1(0) = 0, \quad K_1 = 0, \quad \psi_1(x = L) = 0 = iK_1 \sin\left(\frac{\kappa L}{\hbar}\right), \quad (110)$$

$$\psi_2(0) = 0, \quad K_2 = 0, \quad \psi_2(x = L) = 0 = iK_2 \sin\left(\frac{\tilde{\kappa} L}{\hbar}\right), \quad (111)$$

and, we obtain a wave number into box with infinity walls

$$\kappa_{+/-j} = \frac{\pi \hbar 2j_{+/-}}{2L}. \quad (112)$$

$$K_1 = -\frac{i2}{c\hbar} \sqrt{\frac{k}{m}}, \quad K_2 = -\frac{i2}{c\hbar} \sqrt{\frac{k}{m}}, \quad (113)$$

with  $j_{+/-}$ ] *Relativistic it is instructive to consider  $\kappa_r$  as angular wave-number.* There is a relationship as in (??), and it is necessary to establish

$$\begin{aligned}\kappa_r &= \frac{r\pi}{2(n+1)l_P}, \quad \text{con } r = 1, \dots, N+1. \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin(k_r l_P) \approx c_s k_r \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ \omega_r^2 &= \left(\frac{4k}{m}\right) \sin^2\left(\frac{1}{2}\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ &= 1 - \cos\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right)\end{aligned}\quad (114)$$

with  $c_s = c = 2\sqrt{\frac{k}{m_P}} l_P$ , and  $a = l_P$  is distance of equilibrium points. This above expression is the equivalent to polymeric quantum mechanics with  $c_s$  as speed of light. It is necessary to mention that is possible to obtain

$$k = k_P = m_P \left(\frac{c}{2l_P}\right)^2, \quad (115)$$

that is as very small time-space distortion.

### A. Crystal and Relativistic Wavenumber

it is instructive to consider  $\kappa_r$  as angular wave-number. There is a relationship as in (??), and it is necessary to establish

$$\begin{aligned}\kappa_r &= \frac{r\pi}{2(n+1)l_P}, \quad \text{con } r = 1, \dots, N+1. \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin(k_r l_P) \approx c_s k_r \\ \omega_r &= 2\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ \omega_r^2 &= \left(\frac{4k}{m}\right) \sin^2\left(\frac{1}{2}\sqrt{\frac{m}{k}} c_s \kappa_r\right) \\ &= 1 - \cos\left(\sqrt{\frac{m}{k}} c_s \kappa_r\right)\end{aligned}\quad (116)$$

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$$k = k_P = m_P \left(\frac{c}{2l_P}\right)^2, \quad (117)$$

that is as very small time-space distortion.

## B. Pais-Uhlenbeck and Bernard-Duncan Model

The structure of this simplified crystal has a relationship with highest order formalism that was studied by Pais-Uhlenbeck [27].

To get started, we consider a density Hamiltonian with opposite signs that we viewed is formed by linear oscillators, representing a crystal

$$H = \sum_{r=1}^n \left\{ \frac{1}{4\rho} \pi_{\Psi_r}^2 - (\partial_x \Psi_r)^2 + \frac{\rho \omega_r^2}{4} \psi_r^2 - \left[ \frac{1}{4\rho} \pi_{\Psi_{-r}}^2 - (\partial_x \Psi_{-r})^2 + \frac{\rho \omega_r^2}{4} \psi_{-r}^2 \right] \right\}, \quad (118)$$

and It was studied in [27] where it is proposed a non-local transformation

$$\begin{aligned} \psi_r &= \frac{(\square + \rho \omega_r^2)}{2(2\epsilon \rho \omega_r)^{\frac{1}{2}}} \varphi_r, & \psi_{-r} &= \frac{(\square + (1+2\epsilon)\rho \omega_r^2)}{2(2\epsilon \rho \omega_r)^{\frac{1}{2}}} \varphi_r, \\ \pi_{\Psi_r} &= \frac{(\square + \rho \omega_r^2)}{2((1+2\epsilon)\rho \omega_r)^{\frac{1}{2}}} \dot{\varphi}_r, & \pi_{\Psi_{-r}} &= \frac{(\square + (1+2\epsilon)\rho \omega_r^2)}{2((1+2\epsilon)\rho \omega_r)^{\frac{1}{2}}} \dot{\varphi}_r, \end{aligned} \quad (119)$$

with  $\omega_{2r}^2 = \omega_r^2 + 2\epsilon \omega_r$ . It is important to mention that is included momenta in these transformations.

We note that last model (118) has a non-local relationship with [27], Pais Uhlenbeck model that is named as Bernard Duncan model for fields

$$\mathcal{L}_{BD} = \sum_{r=1}^n \left\{ -\frac{1}{2} (\square \varphi_r)^2 + (1+\epsilon) \rho \omega_r \partial_\mu \varphi_r \partial^\mu \varphi_r - \frac{(1+2\epsilon) \rho^2 \omega_r^4}{2} \varphi_r^2 \right\}, \quad (120)$$

that is Pais-Uhlenbeck model in particles and Bernard-Duncan model for fields.

where we observe that the higher order model has relationship with Klein-Gordon Hamiltonian with opposite sign that produces negative norm states. In this point, we must consider to establish a different form that describes a crystal, in our case linear oscillators, thinking space-time as a crystalline structure [20].

## C. No Canonical Transformation

There is a transformation where it can decouple particles of a crystal by means eigenvalues [33] that in this case is obtained Klein Gordon fields oscillating with different frequencies.

The new disengage system has new wave functions

$$\begin{aligned} \varphi_j(x) &= \sum_{r=1}^n \beta_r a_r \sin\left(j \frac{r\pi}{(n+1)}\right) \Psi_r, \\ \varphi_j^\dagger(x) &= \sum_{r=1}^n \beta_r^\dagger a_r^\dagger \sin\left(j \frac{r\pi}{(n+1)}\right) \Psi_r^\dagger, \\ \pi_{\varphi_j} &= \sum_{r=1}^n \beta_r a_r \sin\left(j \frac{r\pi}{(n+1)}\right) \Pi_r, \\ \pi_{\varphi_j}^\dagger &= \sum_{r=1}^n \beta_r^\dagger a_r^\dagger \sin\left(j \frac{r\pi}{(n+1)}\right) \Pi_r^\dagger, \end{aligned} \quad (121)$$

or in a Hamiltonian Transformation

$$H_{MN} = \sum_{r=-n}^n \frac{1}{2\rho} \Pi_r^2 + \frac{\rho \omega_r^2}{2} \Psi_r^2, \quad (122)$$

with  $k_r = \rho\omega_r^2$ ,  $\Pi_r = \rho\dot{\Psi}_r$  and field  $\Psi_r$  is (121). It is observed a relationship with the harmonic oscillator. These coordinates are not a canonical system, but it is a way of normal modes that we can use to describe in spacetime with polymeric quantum mechanics. It is necessary to mention that with this change we have a positive sign in Hamiltonian, and there is a no-canonical transformation. However, it exists a complex canonical transformation [6, 15, 16].

#### D. Energy Deformation and Space-Time.

It is important to include [?] with Hamiltonian form in path integral. The Lagrangian form is useful, but in our case not permit to note the structure. Then we use (116) and obtain

$$E_{n,r}^2 = \hbar^2 \omega_{n,r}^2 = \Pi_{n,r}^4 c^2 + m^2 c^4, \quad (123)$$

that is the relativistic energy

$$\omega_{n,r}^2 = \sin^2\left(\frac{l_P}{c}\omega_{n,r}\right) = \frac{4k}{m_P}(1 - \cos\left(\frac{m_P}{k}c\kappa_r\right)) \quad (124)$$

$$\approx \frac{c^2}{\hbar^2 \mu_0^2} (1 - \cos\left(\frac{\mu_0}{\hbar}\Pi_r\right))^2 + \frac{m_P^2 c^4}{\hbar^2}, \quad (125)$$

that is the oscillation frequency for space-time, and it is possible to calculate the spring constant for the space-time

$$k \approx \frac{m_P c^2}{4\hbar^2 \mu_0^2} = \sqrt{\frac{\hbar c^5}{16G\mu_0^4}}, \quad (126)$$

that is a deformation in loop quantum mechanics.

#### E. Planck Length

We analyze small scales where it is difficult to establish a measure if it crosses scales of quantum mechanics, and there are conceptual problems and difficult managing.

In order to establish this description, we don't take a limit to  $\mu_0 \rightarrow 0$  instead we take the smallest quantities [3, 4, 21]

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \quad m_P = \sqrt{\frac{\hbar c}{G}} \quad (127)$$

and with wave numbers that is established for this linear crystal We use wave number and it is established a new quantity that define scales and is used in polymeric mechanics (133) as

$$\mu_0 = \frac{\pi}{2(n+1)l_P} = \frac{\pi c^3}{2(n+1)\hbar G}, \quad (128)$$

that in this case is used for a red of little oscillations that represent spacetime structure.

### VII. FRACTAL DIMENSION

It is important to mention as is defined a dimension in a fractal using

$$D = \lim_{\epsilon \rightarrow 0} \frac{\log |S(\epsilon)|}{\log |\frac{1}{\epsilon}|}, \quad (129)$$

that is Hausdorff- Besicovitch dimension and  $S(\epsilon)$  is number of self-similar structures with linear side  $\epsilon$  that is necessary to cover structure.

With this idea is possible to construct a fractal that establish chaos. If we consider

$$\pi_r = \frac{1}{\mu_0} \sin(\mu_0 \pi_r), \quad (130)$$

where is supposed that momentum is complex in general []. Fractal dimension as is proposed by Hausdorff- Besicovitch is .

We use wave number and it is established a new quantity that define scales and is used in polymeric mechanics (133) as

$$\mu_0 = \frac{\pi}{2(n+1)l_P} = \frac{\pi c^3}{2(n+1)\hbar G}, \quad (131)$$

that in this case is used for a red of little oscillations that represent space-time structure.

## VIII. FRACTAL STRUCTURE WITH JULIA METHOD

If it exists a complex structure of canonical transformation in canonical structure [6, 15, 17, 18, 20], we could think in a fractal structure for nature. We think it exist a ordering of nature that forms a fractal structure that is noted in each scale. No confuse particles that there are with structures in universe. It is a way of ordering, no physical object. Thinking in ordering for nature, including space, waste less energy repeat structures by means of fractals, but it is not a fundamental structure because ordering is not physical object. Space-time complies with this rule of minimal ordering that is perceptive where in this ordering matter is more establish.

The Julia set are fractals that are obtained studying iteration of holomorphic function for complex numbers that could present in complex canonical mechanics [15–18] and quantum structure where complex numbers prevalent.

It is possible to define a set called Julia set  $\mathcal{J}$  and it is an holograph could be written as the Julia set that depends of function  $f$  and  $\mathcal{J}(f)$ . The general idea is to begin the complex iteration

$$z = f(z),$$

in this way for example iterating 3 times

$$z = f(f(f(z))),$$

that in terms of  $\sin(x)$  or  $\cos(x)$  and Euler's products [14] could generate an alternative way to Taylor approximate. An important fact is that if  $||z_n|| > 2$ , then succession is divergent and  $||z||$  does not belong to Julia set [12, 13]. Other way of writting this succession is

$$z_{n+1} = f(z_n),$$

In terms of frequencies we think that we have an extended complex quantities, no real quantities as Bender works.

### A. Julia's Fractal and Phonons

In order to describe a complicated structure as a phonon, that is a virtual particle that help to describe a red of particles or properties of space-time that can be described with Planck mass and Planck length with frequencies in a structure, frequency in this case depend of structure.

It is necessary to consider frequency as a structure fractal

$$\omega_r = 2\sqrt{\frac{k}{m}} \sin\left(\frac{a}{c}\omega_r\right), \quad (132)$$

here we include structure of fractal that can result contradictory, but It is supposed that is in different scales correct. In summary, we use this notation in order to construct a fractal Fig. 2. Fractal that we generate is Julia's fractal is more common in lowest velocities.

It is instructive to consider a mechanical model of this chain of oscillators. We consider differences between the closest wave numbers in this model

$$\begin{aligned} \mu_0^{cl} &= \kappa_r - \kappa_{r-1} = \left( \frac{r\pi}{2(n+1)a} - \frac{(r-1)\pi}{2(n+1)a} \right) \\ &= \frac{\pi}{2(n+1)a}, = \frac{\pi}{2l_P}, \end{aligned} \quad (133)$$

that is important to improve this description and realize structures in next quantum model with polymer quantum mechanics.

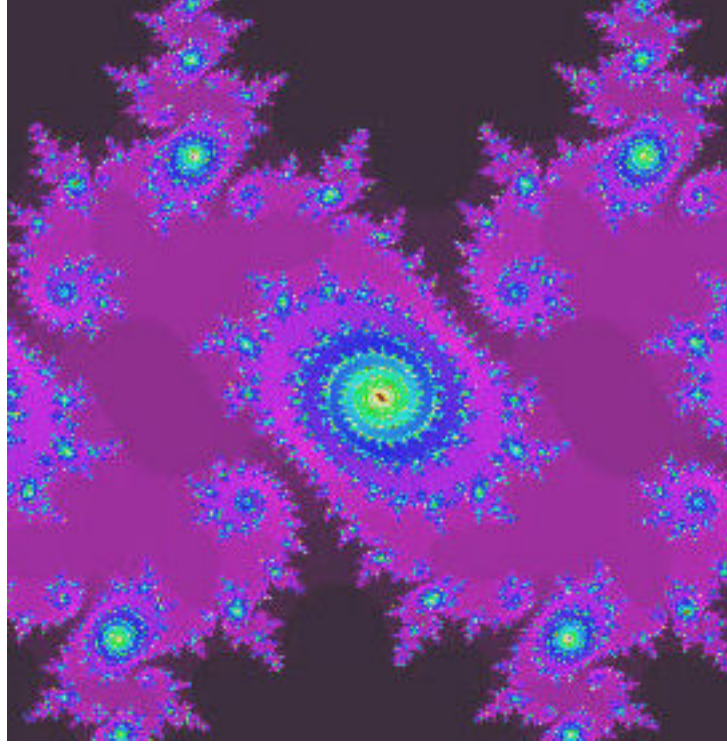


FIG. 2. Julia set of fractal associated to this frequency. The images is from <https://oocities.org/valcocy/sinz.html>

### B. Particle In a Box

Considering a particle in a box with a border in  $x = 0, L$  in quantum mechanics. We consider Schrodinger equation for a free particle

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(t, x), \quad (134)$$

and it is usual for solve this equation

$$\begin{aligned} \Psi(t, x) &= \phi(t)\psi(x) \\ i\hbar \frac{\partial}{\partial t} \phi_n(t) &= E_n \phi_n(t), \end{aligned} \quad (135)$$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n = E_n \psi_n, \text{ with } 0 < x < L \quad (136)$$

with solutions

$$\begin{aligned} \phi_n(t) &= \exp\left(-i \frac{E_n}{\hbar} t\right), \\ \psi_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{\sqrt{2mn}\pi x}{\hbar L}\right), \end{aligned} \quad (137)$$

but this equation has two solutions that describe two possible solutions as Schrodinger equation only consider a part. In this case infinity walls is a hypothetical until case because it is illustrative with opposite direction solutions. The extra opposite solution in a box is

$$\psi_{-n}(x) = \sqrt{\frac{2}{L}} \sin\left(-\frac{\sqrt{2mn}\pi x}{\hbar L}\right), \quad (138)$$

and the first equation (135) has to had a part temporal that evolves

$$\frac{d^2}{dt^2}\phi_n(t) + \frac{c^2 E_n}{\hbar}\phi_n(t) = 0, \quad (139)$$

that in case of this equation, we must consider solution between this two walls for Schrodinger equation where is known

$$[\partial_t - c_s \partial_x - \frac{E_n}{\hbar}][\partial_t + c_s \partial_x + \frac{E_n}{\hbar}]\Psi(t, x) = 0 \quad (140)$$

we agree that is a linear combination from the first case, but in order to consider another complete set solution in this set is

$$\Psi_n = \psi_n + \psi_{-n} = \sqrt{\frac{2}{L}}[\sin(\frac{n\pi}{L}(x - c_s t)) + \sin(-\frac{n\pi}{L}(x + c_s t))]\Theta(t_w - t) \quad (141)$$

that is a solution for a higher order theory []. Total solution is an algebraic solution

$$\Psi_n = \sum_{j=1}^n \psi_j = \sqrt{\frac{2\pi}{L}} \sum_{j=1}^n [\sin(\frac{j\pi}{L}(x - c_s t)) + \sin(-\frac{j\pi}{L}(x + c_s t))] \quad (142)$$

$$= \sqrt{\frac{2}{L}} \left[ \frac{2}{\sin(\frac{(x - c_s t)}{2})} \sin(\frac{(n+1)\pi}{2L}(x - c_s t)) \right] \sin(\frac{n(x - c_s t)}{2}), \quad (143)$$

where we mention second solution is dependent linear of first solution (142) with negative solution for appositve's moving sign that is important to consider, because particles are maintained between infinity walls. Energy for free particles in quantum mechanics is

$$E_r = \frac{p_r^2}{2m} = \frac{\hbar^2 k_r^2}{2m} = \sum_{\kappa=-r}^r E_\kappa = \sum_{\kappa=-r}^r \frac{\hbar^2 \pi^2}{mL^2} n_\kappa^2 = \frac{\hbar^2 \pi^2 (2r+1)}{mL^2} \frac{r(r+1)}{6}, \quad (144)$$

that is a dependent non-linear term including in this base. It is important to mention that we have  $-j$  a return wave that is dependent of wave in the first part. Only walls could absorb energy, but in this point we suppose that this exchange is not possible.

### 1. Klein-Gordon Equation

It is important to consider relativistic energy

$$E^2 = p^2 c^2 + m^2 c^4, \quad (145)$$

where we promote quantum operators to dynamics variables

$$\hat{E} = \hat{p}_0 = i\hbar \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -i\hbar \nabla, \quad (146)$$

$$\frac{\partial^2}{\partial t^2} \Psi(t, x) - c^2 \frac{\partial^2}{\partial x^2} \Psi(t, x) = -\frac{m^2 c^4}{\hbar^2} \Psi(t, x) \quad (147)$$

$$\Psi(t, x) = \phi(t) \psi(x), \quad (148)$$

$$\frac{1}{\phi(t)} \frac{\partial^2}{\partial t^2} \phi(t) = -\left(\frac{E^2}{\hbar^2}\right) \quad (149)$$

$$\frac{1}{\psi(x)} \frac{\partial^2}{\partial x^2} \psi(x) = -\left(\frac{p^2 c^2}{\hbar^2}\right). \quad (150)$$

Solutions for this equations are

$$\phi(t) = e_0 \exp(i \frac{E}{\hbar}(t - t_0)) + e_1 \exp(-i \frac{E}{\hbar}(t - t_0)), \quad (151)$$

$$\psi(x) = P_0 \exp((i \frac{cp}{\hbar}(x - x_0) + \delta_0) + P_1 \exp(-i \frac{cp}{\hbar}(x - x_0) + \delta_1), \quad (152)$$

In a quantum level Klein-Gordon is not feasible option as it produce negative probabilities, but for quantum field theory it exists.

Another hand this solutions for position is

$$\psi(0) = \psi(L) = 0, \quad (153)$$

$$\psi_+(x) = P_0 [\cos(\frac{cp}{\hbar}x + \delta_0) + i \sin(\frac{cp}{\hbar}x + \delta_0)], \quad (154)$$

$$\psi(0) = \cos(\delta_0) + i \sin(\delta_0), \quad \psi(L) = \cos(\frac{cp}{\hbar}L + \delta_0) + i \sin(\frac{cp}{\hbar}L + \delta_0), \quad (155)$$

$$\psi(L) = \cos(\frac{cp}{\hbar}L) \cos(\delta_0) - \sin(\frac{cp}{\hbar}L) \sin(\delta_0) + i \sin(\frac{cp}{\hbar}L) \cos(\delta_0) + i \sin(\delta_0) \cos(\frac{cp}{\hbar}L) \quad (156)$$

$$\psi(L) = \cos(\delta_0) [\cos(\frac{cp}{\hbar}L) + i \sin(\frac{cp}{\hbar}L)] + i \sin(\delta_0) [\cos(\frac{cp}{\hbar}L) + i \sin(\frac{cp}{\hbar}L)] \quad (157)$$

It is important to mention that in the wall when  $x = 0$  then we obtain that  $\delta_0 = 0$

$$\psi_+(L) = [\cos(\frac{cp}{\hbar}L) + i \sin(\frac{cp}{\hbar}L)] = \exp(i \frac{cp}{\hbar}L), \quad \psi_+(0) = 1, \quad (158)$$

and

$$p_j = \frac{\pi \hbar j}{L}, \quad \text{Momentum for particle in a box where it depends of } L \text{ size of box} \quad (159)$$

$$p_j = \frac{\pi \hbar j}{L}, \quad \text{Momentum for a bounced particle that is bounded in a box. This wave front can be altered by second wall. In this model there is not friction and in walls there are not perfect rigid limits.} \quad (160)$$

and total energy including each n-particle in box is

$$\sum_{j=0}^n E_j = Lc^2 \frac{\hbar^2 \pi^2}{2mL^2} \sum_{j=-n}^n j^2 + (n+1)m^2 c^4 = \frac{\hbar^2 \pi^2}{mL} \frac{n(n+1)(2n+1)}{6} c^2 + (n+1)m^2 c^4, \quad (161)$$

where in case of negative movement, we get

$$\psi_-(L) = \cos(\delta_-) [\cos(\frac{\wp}{\hbar}L) - i \sin(\frac{\wp}{\hbar}L)] + i \sin(\delta_-) [\cos(\frac{\wp}{\hbar}L) - i \sin(\frac{\wp}{\hbar}L)] = \exp(-i \frac{\wp}{\hbar}L), \quad \psi_-(L) = 0, \quad (162)$$

and it is established phases  $\delta_- = 0$  and  $\delta_- = 2\pi$

$$\psi_+(x) = \cos(\frac{\pi j_+}{L}x) - i \sin(\frac{\pi j_+}{L}x), \quad (163)$$

$$\psi_-(x) = \cos(\frac{\pi j_-}{L}xi) - i \sin(\frac{\pi j_-}{L}x), \quad (164)$$

with wave zero in walls

$$\psi_+(x) = \sin(\frac{\pi j_+}{L}x), \quad \psi_-(x) = -\sin(\frac{\pi j_-}{L}x), \quad (165)$$

that is total solution of Klein-Gordon equation

$$\Psi(x) = \psi_+(\frac{\pi j_+}{L}x) + \Psi_-(\frac{\pi j_-}{L}x), \quad (166)$$

that of course both solutions are dependent linear, but we do not abother way to say that both waves go in opposite directions.

### C. Relativistic Dispersion

It is usual in modern physics to associate energy with frequency and wave number with momentum. Here we consider a structure in a little crystal, where space-time in small scales is considered with a representation of quantum mechanics according to description. we consider first a relationship between frequency, wave number and relativistic description

$$\omega^2 = \kappa^2 c^2 + \frac{c^4}{\hbar^2} m^2, \quad (\omega - \kappa c)(\omega + \kappa c) = \frac{c^4}{\hbar^2} m^2, \quad (167)$$

$$(\omega - \kappa c) = \frac{c^2}{\hbar} m, \quad (\omega + \kappa c) = \frac{c^2}{\hbar} m, \quad (168)$$

where dispersion relationship has two directions, that include opposite direction.

### D. Phonons In a Box

In order to handle difficult cases, we consider phonons method that is useful in this situation

$$\mathcal{H} = \sum_{n=-r}^r \mathcal{H}_n, \quad (169)$$

$$\mathcal{H}_n = \frac{1}{2m} \pi_n^2 + \frac{m\omega_n^2}{2} \psi_n^2, \quad (170)$$

$$\omega_n = 2\sqrt{\frac{k}{m}} \sin\left(\frac{\pi n}{2L}\right), \quad \omega_n^2 = \frac{4k}{m} (1 - \cos\left(\frac{\pi n}{2L}\right)), \quad (171)$$

where we use harmonic oscillator frequencies

$$\Psi(x) = \prod_{n=-r}^r \psi_n(x), \quad (172)$$

that is applied in quantum mechanics and it is a way of describing a system for a lot of particles.

### E. Phonon Hamiltonian In a Box

we build a system of virtual particles in an usual way with a infinity line of harmonic oscillators  $n \rightarrow \infty$ .

$$\mathcal{H}_n = \frac{1}{2m} \pi_n^2 + \frac{m\omega_0^2 n^2}{2} \psi_n^2, \quad (173)$$

$$L = (N + 1) \approx N \quad \text{If } N \gg 1 \text{ is very huge.} \quad (174)$$

where frequencies are regulated by quantity of particles

$$\omega_n = \sqrt{\frac{k}{m}} \sin\left(\frac{\pi n}{2L}\right) \approx \sqrt{\frac{k}{m}} \left(\frac{\pi n}{2L}\right) \quad \omega_n^2 = \frac{4k}{m} (1 - \cos\left(\frac{\pi n}{2L}\right)), \quad (175)$$

$$\lim_{N \rightarrow \infty} \sin\left(\frac{\pi n}{2L}\right) \approx \frac{\pi n}{2L} \quad \omega_0 = \frac{\pi}{2L} \quad (176)$$

with  $L = N + 1$  and  $\omega_1 = \omega_0, \omega_2 = 2\omega_0, \omega_3 = 3\omega_0, \dots$  We can note

$$\mathcal{H}_n = \left( \frac{1}{2m} \pi_n^2 + \frac{m(c^2 \kappa^2 + \frac{m^2 c^4}{\hbar^2})}{2} \psi_n^2 \right), \quad (177)$$

and wave functions for this system

$$\psi(x) = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi \sqrt{2m}}{\hbar L} x\right) + \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \sin\left(-\frac{\pi \sqrt{2m}}{\hbar L} x\right), \quad (178)$$

$$p = \frac{\pi \sqrt{2m}}{\hbar L}, \quad \wp = -\frac{\pi \sqrt{2m}}{\hbar L}. \quad (179)$$

and as frequencies we handle momentum, this can be most natural

$$p_r = \frac{2\hbar}{c} \sqrt{\frac{k}{m}} \sin\left(\frac{r\pi}{2(n+1)a}\right), \quad (180)$$

but exist an alternative way with higher order theories that describe wave function that bounces in a wall.

## IX. PARTICLE IN A BOX WITH HIGHER ORDER THEORIES

In order to establish a system that includes wave function that bounces, we studies Klein-Gordon equation for free particle in a box

$$\hat{E}\phi(t) = (c\kappa + \frac{mc^2}{\hbar})\phi(t), \quad \hat{E}\phi(t) = (\frac{mc^2}{\hbar} - c\kappa)\phi(t), \quad (181)$$

$$(i\frac{\partial}{\partial t} - c\tilde{\kappa})\phi_1(t) = \frac{c^2}{\hbar}m\phi_1(t), \quad (i\frac{\partial}{\partial t} + c\kappa)\phi_2(t) = \frac{c^2}{\hbar}m\phi_2(t), \quad (182)$$

$$\tilde{\kappa} = \kappa_{-,j} = \frac{\pi j_-}{L}, \quad \kappa_{+,j} = \frac{\pi j_+}{L}, \quad (183)$$

with wave functions

$$\phi_1(t) = \alpha_1 \exp\left(-\frac{it(mc^2 + c\hbar\tilde{\kappa})}{\hbar}\right), \quad \phi_2(t) = \alpha_2 \exp\left(-\frac{it(mc^2 - c\hbar\kappa)}{\hbar}\right), \quad (184)$$

that no-necessary are unitaries.

$$(E - \frac{c\hbar}{i}\frac{\partial}{\partial x})\psi_1(x) = c^2m\psi_1(x), \quad (E + \frac{c\hbar}{i}\frac{\partial}{\partial x})\psi_2(x) = c^2m\psi_2(x),$$

$$\psi_1(x) = k_1 \cos\left(\left[\frac{E_- - mc^2}{\hbar c}\right]x\right) + ik_1 \sin\left(\left[\frac{E_- - mc^2}{\hbar c}\right]x\right), \quad (185)$$

$$\psi_2(x) = k_2 \cos\left(\left[\frac{E_+ - mc^2}{\hbar c}\right]x\right) - ik_2 \sin\left(\left[\frac{E_+ - mc^2}{\hbar c}\right]x\right), \quad (186)$$

with frequencies

$$\omega_- = mc^2 + \hbar c\kappa_-, \quad \omega_+ = mc^2 - \hbar c\tilde{\kappa}_+,$$

$$\omega_- = \frac{mc^2}{\hbar} + c\kappa, \quad \omega_+ = \frac{mc^2}{\hbar} - c\tilde{\kappa}, \quad (187)$$

with boundaring conditons

$$\psi_1(0) = 0, \quad K_1 = 0, \quad \psi_1(x=L) = 0 = iK_1 \sin\left(\frac{\kappa L}{\hbar}\right), \quad (188)$$

$$\psi_2(0) = 0, \quad K_2 = 0, \quad \psi_2(x=L) = 0 = iK_2 \sin\left(\frac{\tilde{\kappa} L}{\hbar}\right), \quad (189)$$

and, we obtain a wave number into box with infinity walls

$$\kappa_{+/-j} = \frac{\pi \hbar 2j_{+/-}}{2L}. \quad (190)$$

$$K_1 = -\frac{i2}{c\hbar} \sqrt{\frac{k}{m}}, \quad K_2 = -\frac{i2}{c\hbar} \sqrt{\frac{k}{m}}, \quad (191)$$

with  $j_{+/-} = -n, \dots, -1, 0, 1, \dots, n$ .

In other hand, a system that has a linear crystal of harmonic oscillators

$$\kappa_{-,r} = -\frac{2}{\hbar c} \sqrt{\frac{k}{m}} \sin\left(\frac{\pi \hbar r_+}{2L}\right), \quad \kappa_{+,r} = \frac{2}{\hbar c} \sqrt{\frac{k}{m}} \sin\left(\frac{\pi \hbar r_+}{2L}\right), \quad (192)$$

$$\tilde{\kappa}_r = \kappa_{-,r} = -\frac{2\omega}{\hbar c} \sin\left(\frac{p_{+,r}}{2}\right), \quad \kappa_{+,r} = \frac{2\omega}{\hbar c} \sin\left(\frac{p_{+,r}}{2}\right), \quad (193)$$

where  $r_- = -r_+$  that is similar to a crystal structure. We can establish a relationship with Pais-Uhlenbeck model that is higher order model with a non-canonical transformation because model has order dynamics. Transformation is

$$\begin{aligned}\psi_{+,r}(\kappa) &= \frac{(\Pi_{-r}(\kappa) + \omega_{-r}^2 \Phi_r(\kappa))}{[\omega_{-r}^2 - \omega_r^2]^{\frac{1}{2}}}, & \psi_{-,r}(\kappa) &= \frac{(\Pi_r(\kappa) + \omega_{-r}^2 \Phi_{-r}(\kappa))}{\omega_r[\kappa_{-r}^2 - \omega_r^2]^{\frac{1}{2}}} \\ \pi_{-r}(\kappa) &= \frac{\kappa_r^2(\Pi_{-r}(\kappa) + \omega_r^2 \Phi_r)}{[\omega_r^2 - \omega_{-r}^2]^{\frac{1}{2}}}, & \pi_r(\kappa) &= \frac{(\Pi_r(\kappa) + \omega_r^2 \Phi_{-r}(\kappa))}{[\omega_{-r}^2 - \omega_r^2]^{\frac{1}{2}}},\end{aligned}\quad (194)$$

with this transformation we obtain next Hamiltonian

$$\mathcal{H}_{PU} = \sum_{r=1}^n \frac{\Pi_{-r}^2}{2} + \Phi_{-r} \Pi_r + \frac{(\kappa_{-r}^2 + \kappa_r^2 + 2m^2 c^4)}{2} \Phi_{-r}^2 - \frac{(mc^2 + \hbar c \tilde{\kappa}_-)^2 (mc^2 - \hbar c \tilde{\kappa}_+)^2}{2} \Phi_r, \quad (195)$$

that result from Lagrangian density

$$\mathcal{L} = \sum_{r=1}^n \frac{1}{2} [\ddot{\Phi}_r^2 + (\kappa_r^2 + \kappa_{-r}^2 + 2m^2 c^4) \dot{\Phi}_r + (mc^2 + \hbar c \tilde{\kappa}_-)^2 (mc^2 - \hbar c \tilde{\kappa}_+)^2 \Phi_r], \quad (196)$$

with  $\tilde{\kappa}_r = \kappa_{-r}$ .

In other hand, in a box of this free particle model. We consider differences between the closest wave numbers in this model

$$\mu_0^+ = \mu_0^- = \kappa_r - \kappa_{r-1} = \left( \frac{r^2 \pi^2}{2L} - \frac{(r-1)^2 \pi^2}{2L} \right) = \frac{\pi}{L}, \quad (197)$$

$$\sum_{j=0}^n E_j = \sum_{j=0}^n \frac{\hbar^2 \pi^2}{2mL^2} j^2 L c^2 + (n+1)m^2 c^4 = \frac{\hbar^2 \pi^2}{2mL} \frac{n(n+1)(2n+1)}{6} c^2 + (n+1)m^2 c^4, \quad (198)$$

that is important to improve this description and realize structures in next quantum model with polymer quantum mechanics.

### A. Quantum Particle In a Box

In other hand, for a free particle in a box that bounced

$$i\hbar \frac{\partial^2}{\partial t^2} \Psi(t, x) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(t, x), \quad (199)$$

and solve this equation  $\Psi(t, x) = \phi(t)\psi(x)$

$$\begin{aligned}\frac{i\hbar}{\phi_r(t)} \frac{\partial^2}{\partial t^2} \phi_r(t) &= r\omega_1 \\ r\omega_1 &= -\frac{1}{\psi_r(x)} \frac{\hbar^2}{2m} \nabla^2 \psi_r(x),\end{aligned}\quad (200)$$

that consider frequencies

$$\phi_r(t) = e^{\sqrt{\frac{r\omega_1}{2\hbar}} t} e^{-i\sqrt{\frac{r\omega_1}{2\hbar}} t}, \quad \psi(x) = e^{i\sqrt{\frac{2mr\omega_1}{\hbar}} x} \quad (201)$$

that can be obtained solving wave equation noting that exist two solutions with  $(x - vt)$  and  $(x + vt)$ .

## B. Holonomy and Polymeric Quantum Mechanics

We analyze how to establish a method for little scales. In this case is used polymer quantum mechanics that is based in translation operators. In order to develop holonomy, we establish a diagonal Hamiltonian a canonical transformation from line of oscillators. that separates in oscillators equivalent to a chain of oscillators coupled. There is a quantum representation, Witt representation or polymeric quantum mechanics [21, 31].

$$\pi_{\kappa_r} = \pi_r = \frac{\hbar}{\mu_0} \sin\left(\frac{\mu_0 \pi_r}{\hbar}\right) \quad (202)$$

and as quantum mechanics

$$\pi_r^2 = \frac{2\hbar^2}{\mu_0^2} (1 - \cos(\frac{\mu_0 \pi_r}{\hbar})), \quad (203)$$

that in a crystalline structure, in special for this chain of oscillators

$$[\varphi(\kappa), \pi(\lambda)] = i2\pi\hbar\delta(\kappa + \lambda), \quad (204)$$

$$\begin{aligned} U(\lambda)\hat{\varphi}_{\kappa_r} &= \exp(i\lambda\hat{\varphi}_{\kappa_r})\hat{\varphi}_{\kappa_r}, \\ V(\mu)\hat{\varphi}_{\kappa_r} &= \exp(\frac{i}{\hbar}\mu\hat{\pi}_{\kappa_r})\hat{\varphi}_{\kappa_r} = \hat{\varphi}_{\kappa_r - \mu}, \end{aligned} \quad (205)$$

## C. Density Hamiltonian and Polymeric Mechanics

It possible obtain a Hamiltonian in terms of wave number that is discrete quantities that are resulted from this model. We obtain wave numbers that depend of all system, that is strongly depended of border conditions. In a nutshell, it is a knew system, because we meet that is a set of harmonic oscillators [21, 25, 26].

Hamiltonian density with Fourier transformation in order to apply polymeric transformation is

$$\begin{aligned} H = \sum_{r=1}^n & \left[ -\frac{1}{2\rho} \frac{\hbar^2}{\mu_0^2} \left( \sin\left(\frac{\mu_0 \pi_{\kappa_r}}{\hbar}\right) \sin\left(\frac{\mu_0 \pi_{-\kappa_r}}{\hbar}\right) \right. \right. \\ & \left. \left. + \frac{\rho\omega_r^2}{2} \varphi_{\kappa_r} \varphi_{-\kappa_r} \right] \right], \end{aligned} \quad (206)$$

and total transformation is

$$\begin{aligned} H_{MN} = \sum_{r=1}^n & \left[ -\frac{1}{4\rho} \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \pi_{\psi_{\kappa_r}}\right) + \frac{\rho\omega_r^2}{2} \psi_{\kappa_r}^2 \right] - \\ & \sum_{r=1}^n \left[ -\frac{1}{4\rho} \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \pi_{\psi_{-\kappa_r}}\right) + \frac{\rho\omega_r^2}{2} \psi_{-\kappa_r}^2 \right], \end{aligned} \quad (207)$$

where it is clear that doing this canonical transformation, we obtain a negative energy that it could result confusing in order to do a quantization, and in classical mechanics is due existence of wave number negative. However we have a transformation non-canonical, but it is correct because is disengage line of oscillators.

It is important to mention that we obtain negative energy and it is fixed that negative energy no is in a direct relationship with positive energy [27] by means of a term of interaction, and complex canonical transformation [6] describe these systems, but here we consider negative no-interacting energy, because the system no describe a only energy, rather a system of particle that travels in opposite directions in a collective movement of particles. In other words, we describe a movement collective, it is not a only direction. The existence of a movement on one direction imply the movement on oppositive direction in a collective of masses.

### D. Euler's expansion of powers

On the other hand, we apply Euler product development instead of establishing a Taylor series expansion. Euler product development is

$$\begin{aligned}\sin(\pi z) &= \pi z \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right) = \pi z \left(1 - \frac{z^2}{1^2}\right) \left(1 - \frac{z^2}{2^2}\right) \dots \approx \pi z - \pi z^3, \\ \cos(\pi z) &= \prod_{m=1}^{\infty} \left(1 - \frac{4z^2}{(2m-1)^2}\right) = \left(1 - \frac{4z^2}{1^2}\right) \left(1 - \frac{4z^2}{3^2}\right) \dots \approx 1 - \left(\frac{1}{1^2} + \frac{1}{3^2}\right) 4z^2 + \frac{16}{3^2} z^4\end{aligned}\quad (208)$$

where is supposed a complex momentum [15, 17, 18] and in polymeric quantum mechanics we get

$$\frac{1}{\mu_0} \sin(\mu_0 \Pi_{q,t}) \approx \frac{1}{\pi} \Pi_{q,t} - \frac{\mu_0^2}{\pi^3} \Pi_{q,t}^3 \quad \frac{1}{\mu_0^2} \cos\left(2 \frac{\mu_0 \Pi_{q,t}}{\pi}\right) \approx 1 - \frac{40}{9} \frac{\mu_0^2 \Pi_{q,t}^2}{\pi^2} + \frac{16 \mu_0^4 \Pi_{q,t}^4}{9 \pi^4} \quad (209)$$

$$\begin{aligned}< \Pi_{out} t_{out} | \Pi_{in}, t_{in} > \approx \mu_0 \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^{\infty}) \\ &\sum_{r=1}^{out+1} \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{N,r}}{(2\pi)^{M_N}} \dots \prod_{r=1}^{M_2} \frac{d\Pi_{2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{1,r}}{(2\pi)^{M_1}} \prod_{r=1}^{M_N} \frac{d\Pi_{-N,r}}{(2\pi)^{M_N}} \dots \prod_{r=1}^{M_2} \frac{d\Pi_{-2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{-1,r}}{(2\pi)^{M_1}} \\ &\exp - \frac{i}{\hbar} \sum_{r=1}^N \left( \sum_{s=1}^{M-1} \left[ \frac{1}{2\rho} \left( 1 - \frac{40}{9} \frac{\mu_0^2 \Pi_{q,t}^2}{\pi^2} + \frac{16 \mu_0^4 \Pi_{q,t}^4}{9 \pi^4} \right) + \frac{\rho \omega_s^2}{2} \Psi_{\kappa,r,s}^2 \right] \right. \\ &\quad \left. + \left[ \frac{1}{2\rho} \left( 1 - \frac{40}{9} \frac{\mu_0^2 \Pi_{r,-s}^2}{\pi^2} + \frac{16 \mu_0^4 \Pi_{r,-s}^4}{9 \pi^4} \right) + \frac{\rho \omega_r^2}{2} \Psi_{\kappa,-s,r}^2 \right] \right),\end{aligned}\quad (210)$$

It is important to mention  $c$  is the light velocity in this type of scales and play as group velocity  $c_s$  in a structure of one-dimension crystal. The energy of harmonic oscillator never is zero.

we should also talk about an approximation of  $\varphi^4$  could be applied in  $\Pi$ , and it could be analyzed [22]

$$\begin{aligned}< \Pi_{out} t_{out} | \Pi_{in}, t_{in} > \approx \mu_0 \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^{\infty}) \\ &\sum_{r=1}^{out+1} \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{N,r}}{(2\pi)^{M_N}} \dots \prod_{r=1}^{M_2} \frac{d\Pi_{2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{1,r}}{(2\pi)^{M_1}} \prod_{r=1}^{M_N} \frac{d\Pi_{-N,r}}{(2\pi)^{M_N}} \dots \prod_{r=1}^{M_2} \frac{d\Pi_{-2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{-1,r}}{(2\pi)^{M_1}} \\ &\frac{16 \mu_0^4}{9 \pi^4} \left[ - \frac{i}{\hbar} \sum_{r=1}^N \sum_{s=1}^{M-1} \left( \frac{\delta^4}{\delta \Pi_{q,t}^4} + \frac{\delta^4}{\delta \Pi_{r,-s}^4} \right) \right] \exp - \frac{i}{\hbar} \sum_{r=1}^N \left( \sum_{s=1}^{M-1} \left[ \frac{1}{2\rho} \left( 1 - \frac{40}{9} \frac{\mu_0^2 \Pi_{q,t}^2}{\pi^2} \right) + \frac{\rho \omega_s^2}{2} \Psi_{\kappa,r,s}^2 \right] \right. \\ &\quad \left. + \left[ \frac{1}{2\rho} \left( 1 - \frac{40}{9} \frac{\mu_0^2 \Pi_{r,-s}^2}{\pi^2} \right) + \frac{\rho \omega_r^2}{2} \Psi_{\kappa,-s,r}^2 \right] \right),\end{aligned}\quad (211)$$

In the next section, a complex transformation is used to front-facing a difficult problem that is a crystal in more dimensions.

### X. FEYNMAN POLYMERIC PATH INTEGRAL

Feynman path integral is a sum of each road that join different two points, it must be consider with this notation that have sums and products that define all road possible in order to establish a dispersion

$$\begin{aligned}< \Pi_{Out}, t_{Out} | \Pi_{In}, t_{In} > = (\exp \{ \frac{iN}{\hbar} \Delta((n-1)\Pi_{n-1} - (1-n)\Pi_{1-n}) \} | \Pi_{In}^{Out} ) (\mu_0^{i-1} \sum_{r=0}^{n-1} \int \prod_{i=2}^n \frac{d\Pi_i}{(2\pi)}) \\ &\exp \{ \frac{i\Delta t}{\hbar} \sum_{r=1}^{n-1} (-r\dot{\Pi}_r + r\dot{\Pi}_{-r} - (\frac{1}{2\rho} \Pi_r^2 + \frac{\rho \omega_r^2}{2} (\mu r)^2) + [\frac{1}{2\rho} \Pi_{-r}^2 + \frac{\rho \omega_{-r}^2}{2} (-\mu r)^2]) \} \}^N\end{aligned}\quad (212)$$

818  $\Delta \mathbf{t} = (\mathbf{t}_{j-1} - \mathbf{t}_{j-2})$ . We consider trajectory before of bouncing sum is done in this path integral where for each  
 819 momentum is assigned all positions and is aggregated the generating of this canonical transformation taking advantage  
 820 path integral formalism

$$821 \quad \langle \Pi_{\text{Out}}, \mathbf{t}_{\text{Out}} | \Pi_{\text{In}}, \mathbf{t}_{\text{In}} \rangle = (\exp\{\frac{i\mathbf{N}}{\hbar}(\Delta(\mathbf{n}-1)\Pi_{\mathbf{n}-1})\}) \left( \int \prod_{i=2}^n d\mathbf{x}^{i-1} \frac{d\Pi_i}{(2\pi)} \sum_{r=0}^{n-1} \exp\{\frac{i\Delta \mathbf{t}}{\hbar} \sum_{r=0}^{n-1} (-\mathbf{r}\dot{\Pi}_r - \mathcal{H})\} \right)^N, \quad (213)$$

822 with Hamiltonian path integral

$$823 \quad \omega_j = \omega(\varphi(x_{j-1}), t_{j-1}, \varphi(x_{j-2}), t_{j-2}) = d\varphi_{j-1} \frac{d\Pi_j}{(2\pi)} \exp\{i(\Pi\dot{\varphi} - \mathcal{H}(\varphi, \Pi))\}, \quad (214)$$

824 that in this notation is included differential elements that is aggregated in polymeric quantum mechanics

$$825 \quad \omega_j = \omega(\varphi(x_j), t_j, \varphi(x_{j-1}), t_{j-1}) = \mu_0 \frac{d\Pi_j}{(2\pi)} \exp\{i\frac{\Delta \mathbf{t}}{\hbar}(\Pi\dot{\varphi} - \mathcal{H}(\varphi, \Pi))\}, \quad (215)$$

826 and  $\mu_0 = d\varphi$  that is a regular partition. In other hand, it is displayed that for this representation exists a quantity  
 827 small. Here we observe that using this notation is obtained

$$828 \quad \langle \Pi_{\text{out},N} \dots \Pi_{\text{out},2}, \Pi_{\text{out},1}, t_{\text{out}} | \Pi_{\text{in},N} \dots \Pi_{\text{in},2}, \Pi_{\text{in},1}, t_{\text{in}} \rangle = \sum_{r_1}^n \dots \sum_{r_N}^n \int_{I_n}^{\text{Out}} \prod_{j=1}^n \omega_j. \quad (216)$$

829 The path integral is developed as Feynman establish [22] and it is natural to make a quantum method. For polymeric  
 830 quantum mechanic it is a starting point that we consider

831 It is necessary to note that  $\mu_r = \mu_0$ , but is a general non-regular partition. In another hand, if the partition is  
 832 regular, we establish

$$833 \quad \langle \Pi_{\text{out},N} \dots \Pi_{\text{out},2}, \Pi_{\text{out},1}, t_{\text{out}} | \Pi_{\text{in},N} \dots \Pi_{\text{in},2}, \Pi_{\text{in},1}, t_{\text{in}} \rangle =$$

$$834 \quad (\mu_0)^N \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}]|_{I_n}^{\text{Out}}) \sum_{r_1=1}^{n-1} \dots \sum_{r_{N-1}=1}^{n-1} \int_{-\infty}^{\infty} \prod_{r_j=1}^n \frac{\omega_{r_j}}{\mu_0}, \quad (217)$$

835 where the Hamiltonian  $\mathcal{H}_{MN}$  is given by (207) and the other side  $N$  is large amount, but  $n$  is partition limit that we  
 836 had done.

837 It is instructive for development to separate a term that depends of total time derivative

$$838 \quad \langle \Pi_{\text{out},N} \dots \Pi_{\text{out},2}, \Pi_{\text{out},1}, t_{\text{out}} | \Pi_{\text{in},N} \dots \Pi_{\text{in},2}, \Pi_{\text{in},1}, t_{\text{in}} \rangle =$$

$$839 \quad (\mu_0)^N \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}]|_{I_n}^{\text{Out}}) \sum_{r_1=1}^{n-1} \dots \sum_{r_{N-1}=1}^{n-1} \int_{-\infty}^{\infty} \prod_{r_j=1}^n \frac{\omega_{r_j}}{\mu_0} \quad (218)$$

840 and it is separated part of path integral that depend under position and it has one part limit in  $\mu_0$  with this  
 841 representation

$$842 \quad \langle \Pi_{\text{out},N} \dots \Pi_{\text{out},2}, \Pi_{\text{out},1}, t_{\text{out}} | \Pi_{\text{in},N} \dots \Pi_{\text{in},2}, \Pi_{\text{in},1}, t_{\text{in}} \rangle = (\mu_0)^n \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}]|_{I_n}^{\text{Out}})$$

$$843 \quad \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta \mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-\Psi_{\kappa,r}\dot{\Pi}_r - \frac{1}{2\rho} \Pi_r^2 + \frac{\rho\omega_r^2}{2} \Psi_{\kappa,r}^2]) \right]^N$$

$$844 \quad \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_{-1}}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta \mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [\psi_{-\kappa,r}\dot{\Pi}_{-r} + [\frac{1}{2\rho} \Pi_{-r}^2 - \frac{\rho\omega_r^2}{2} \Psi_{\kappa,-r}^2]]) \right]^N$$

$$845 \quad \sum_{r=1}^{n-1} r^2 = \frac{1}{6}(n-1)n(2n-1), \quad \sum_{r=1}^{n-1} ar = \frac{an(n-1)}{2}$$

$$846 \quad \zeta_3(0, e^{-\pi}) = \sum_{r=-n}^n \exp(-\pi n^2) = \frac{\sqrt[4]{\pi}}{\Gamma(\frac{3}{4})}, \quad \sum_{n=2}^{\infty} e^{an^2} = \frac{1}{2}[\zeta_3(0, e^{-\pi}) - 2e^a - 1]$$

$$847 \quad \sum_{n=2}^{\infty} e^{an^2} = \frac{1}{2}[\zeta_3(0, e^{-\pi}) - 2e^a - 1], \quad (219)$$

that is Jacobi function theta jacobi and  $n \rightarrow \infty$ . In momenta with position, with a minimum limit in path integral for this representation

$$\zeta_3(\frac{1}{2}, 1) = \sum_{r=-n}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \dot{\Pi}_r r) = \lim_{n \rightarrow \infty} 2 \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \dot{\Pi}_r) \left[ \frac{1 - \exp(-\frac{i\Delta\mathbf{t}}{\hbar} n \dot{\Pi}_r)}{1 - \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \dot{\Pi}_r)} \right], \quad (220)$$

$$\begin{aligned} \zeta_3(\frac{1}{2}, 1) &= \lim_{n \rightarrow \infty} 2 \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \dot{\Pi}_r) \left( \sum_{j=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} (n-j-1) \dot{\Pi}_r) \right) \\ &= \lim_{n \rightarrow \infty} 2 \left( \sum_{j=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} (n-j) \dot{\Pi}_r) \right), \end{aligned} \quad (221)$$

and, if we aggregate this considerations for path integral

$$\begin{aligned} &< \Pi_{out,N} \dots \Pi_{out,2}, \Pi_{out,1}, t_{out} | \Pi_{in,N} \dots \Pi_{in,2}, \Pi_{in,1}, t_{in} > = \\ &= (2\mu_0)^{nN} \exp - \frac{iN}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}] |_{In}^{Out}) \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n-r) \dot{\Pi}_r - \frac{1}{2\rho} \Pi_r^2]) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \sum_{r=1}^{n-1} \exp(\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2}{2\hbar} \Psi_{\kappa_r, \tau}^2) \right]^N \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{\tau=1}^n \frac{d\Pi_{-1}}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n+r) \dot{\Pi}_{-r} - \frac{1}{2\rho} \Pi_{-r}^2]) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \sum_{r=1}^{n-1} \exp(\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2}{2\hbar} \Psi_{\kappa_r, \tau}^2) \right]^N, \end{aligned} \quad (222)$$

that it already consider bounced of phonons particles and  $d^2 = \frac{\hbar}{m\omega_r}$ ,  $m = \frac{\rho}{L}$ , because is linear. The discretized potential is  $U = \frac{\hbar\omega_r}{2d^2} \Psi_{\kappa}^2$ .

$$\begin{aligned} &< \Pi_{out,N} \dots \Pi_{out,2}, \Pi_{out,1}, t_{out} | \Pi_{in,N} \dots \Pi_{in,2}, \Pi_{in,1}, t_{in} > = (2\mu_0)^n \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}] |_{In}^{Out}) \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n-r) \dot{\Pi}_r - \frac{1}{2\rho} \Pi_r^2]) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \sum_{r=1}^{n-1} \exp(\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2}{2\hbar} \Psi_{\kappa_r, \tau}^2) \right]^N \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{\tau=1}^n \frac{d\Pi_{-1}}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n+r) \dot{\Pi}_{-r} - \frac{1}{2\rho} \Pi_{-r}^2]) \right]^N \left[ \prod_{r=1}^{n-1} \sum_{r=1}^{n-1} \exp(\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2}{2\hbar} \Psi_{\kappa_r, \tau}^2) \right]^N, \end{aligned} \quad (223)$$

In other hand, path integral definition based in usual path integral

$$\begin{aligned} &< \Pi_{out} t_{out} | \Pi_{in}, t_{in} > = (2\mu_0)^{nN} \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}] |_{In}^{Out}) \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n-r) \dot{\Pi}_r - \frac{1}{2\rho} \Pi_r^2]) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2(\kappa_r)}{2\hbar}} - 1] \right]^N \\ &\int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \dots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{\tau=1}^n \frac{d\Pi_{-1}}{(2\pi)} \left[ \frac{1}{2} \sum_{r=1}^{n-1} \exp(-\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n+r) \dot{\Pi}_{-r} - \frac{1}{2\rho} \Pi_{-r}^2]) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t} \rho \omega_{\tau}^2(\kappa_r)}{2\hbar}} - 1] \right]^N \end{aligned} \quad (224)$$

and sum infity is

$$\prod_{n=1}^N e^{-\frac{i\Delta\mathbf{t} \rho}{\hbar} \frac{4k}{m} (1 - \cos(\frac{\pi n}{2L}))} = \exp \left\{ -\frac{i\Delta\mathbf{t} \rho}{\hbar} \frac{4k}{m} \left[ \sum_{n=1}^N (1 - \cos(\frac{\pi n}{2L})) \right] \right\} = \exp \left\{ -\frac{i\Delta\mathbf{t} \rho}{\hbar} \frac{4k}{m} \left[ (N - \sum_{n=1}^N \cos(\frac{\pi n}{2L})) \right] \right\}, \quad (225)$$

$$\sum_{n=1}^N \cos(\frac{1}{2L} \pi n) = \frac{1}{2} [-\cos(\frac{\pi}{4L} (N+1)) + \cot(\frac{\pi}{4L}) \sin(\frac{\pi}{2L} (N+1)) - 1],$$

$$\prod_{n=1}^N e^{-\frac{i\rho}{\hbar} \frac{4k}{m} (1 - \cos(\frac{\pi n}{2L}))} = \exp \left\{ -\frac{i\rho}{\hbar} \frac{4k}{m} \left[ (N - \frac{1}{2} [-\cos(\frac{\pi}{4L} (N+1)) + \cot(\frac{\pi}{4L}) \sin(\frac{\pi}{2L} (N+1)) - 1] \right] \right\}, \quad (226)$$

$$\begin{aligned}
& \langle \Pi_{out}, t_{out} | \Pi_{in}, t_{in} \rangle = (2\mu_0)^{nN} \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}] |_{In}^{Out}) \\
& \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \left[ \sum_{r=1}^{n-1} \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [(r-n)\dot{\Pi}_r - \frac{1}{2\rho} \frac{1}{2} (1 - \cos(\Pi_r))] \right) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1] \right]^N \\
& \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_{-1}}{(2\pi)} \left[ \frac{1}{2} \sum_{r=1}^{n-1} \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [-(n+r)\dot{\Pi}_{-r} - \frac{1}{2\rho} \frac{1}{2} (1 - \cos(\Pi_{-r}))] \right) \right]^N \\
& \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1]^N
\end{aligned} \tag{227}$$

and we consider a long chain  $\lim_{N \rightarrow \infty} \omega_n = n\omega_1$ , with  $\omega_\kappa = c\kappa$ ,  
The path integral is

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = (2\mu_0)^{nN} \exp \left( -\frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}] |_{In}^{Out}) \right) \left[ \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \right. \\
& \left. (1/2) \sum_{r=1}^{n-1} \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} [(r-n)\dot{\Pi}_r] \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \frac{1}{4\rho} \sum_{r=1}^{n-1} (1 - \cos \Pi_r) \right) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1] \right]^N \right. \\
& \left. \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_{-1}}{(2\pi)} \right. \\
& \left. (1/2) \left[ \sum_{r=1}^{n-1} \exp \left( \frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} (n+r)\dot{\Pi}_{-r} \right) \left[ \exp - \frac{i\Delta\mathbf{t}}{4\hbar\rho} (1-n) \right] \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \frac{1}{4\rho} (n-1 - [\cos(\frac{\Pi_{(n+1)}{2})) \sin(\frac{\Pi_n}{2}) \frac{1}{\sin(\frac{\Pi_1}{2})}] \right) \right]^N \right. \\
& \left. \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1]^N \right]
\end{aligned} \tag{228}$$

where we have separated part of path integral that depend under the position that has one part limit in  $\mu_0$ . It is important consider that in this system momenta is conserved

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = (2\mu_0)^{nN} \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}] |_{In}^{Out}) \right) \left[ \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_N}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_2}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_1}{(2\pi)} \right. \\
& \left. (\mu_0/2) \left[ \sum_{r=1}^{n-1} \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \sum_{r=1}^{n-1} (r-n)\dot{\Pi}_r \right) \exp \left( -\frac{i}{\hbar} \frac{1}{4\rho} (n-1) \right) \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \frac{1}{4\rho} \sum_{r=1}^{n-1} (-\cos \Pi_r) \right) \right]^N \left[ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) \right. \right. \\
& \left. \left. - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1] \right]^N \int_{-\infty}^{\infty} \prod_{r=1}^n \frac{d\Pi_{-N}}{(2\pi)} \cdots \prod_{r=1}^n \frac{d\Pi_{-2}}{(2\pi)} \prod_{r=1}^n \frac{d\Pi_{-1}}{(2\pi)} \right. \\
& \left. (\mu_0/2) \left[ \sum_{r=1}^{n-1} \exp \left( \frac{i}{\hbar} \sum_{r=1}^{n-1} (n+r)\dot{\Pi}_{-r} \right) \left[ \exp - \frac{i}{4\hbar\rho} (1-n) \right] \exp \left( -\frac{i\Delta\mathbf{t}}{\hbar} \frac{1}{4\rho} \left( \sum_{r=-1}^{1-n} -[\cos(\frac{\Pi_{(r+1)}{2})) \sin(\frac{\Pi_n}{2}) \frac{1}{\sin(\frac{\Pi_1}{2})}] \right) \right]^N \left\{ \prod_{\kappa_r=1}^{n-1} \frac{1}{2} [\zeta_3(0, e^{-\pi}) \right. \right. \\
& \left. \left. - 2e^{-\frac{i\Delta\mathbf{t}\rho\omega_r^2(\kappa_r)}{2\hbar}} - 1] \right\}^N \right]
\end{aligned} \tag{229}$$

with  $\sin(\frac{\Pi_n}{2}) \frac{1}{\sin(\frac{\Pi_1}{2})} = 1$  that is equal for each momenta.

The pathintegral is

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = \mu_0 \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r}\Pi_r - \Psi_{-\kappa,-r}\Pi_{-r}] |_0^\infty) \\
& \sum_{r=1}^{out+1} \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{1,r}}{(2\pi)^{M_1}} \prod_{r=1}^{M_N} \frac{d\Pi_{-N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{-2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{-1,r}}{(2\pi)^{M_1}} \\
& \exp - \frac{i}{\hbar} \sum_{r=1}^N \left( \sum_{s=1}^{M-1} \left[ -\frac{1}{2\rho} \Pi_{r,s}^2 - \frac{\rho\omega_s^2}{2} \Psi_{\kappa,r,s}^2 \right] - \left[ -\frac{1}{2\rho} \Pi_{r,-s}^2 - \frac{\rho\omega_r^2}{2} \Psi_{\kappa,-s,r}^2 \right] \right).
\end{aligned} \tag{230}$$

If we want to use total polymeric quantum mechanics, it is necessary to consider establish momentum for polymeric quantum mechanics and apply Gauss sum quadratic

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = \mu_0^{N-1} \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^\infty) \\
& \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{1,r}}{(2\pi)^{M_1}} \prod_{r=1}^{M_N} \frac{d\Pi_{-N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{-2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{-1,r}}{(2\pi)^{M_1}} \\
& \prod_{r=1}^{out+1} \sum_{s=1}^{N-1} \exp - \frac{i}{\hbar} \left( \sum_{r=1}^{out+1} - \left[ \frac{1}{2\rho} \Pi_{r,s}^2 + \frac{\rho\omega_s^2}{2} \Psi_{\kappa,r,s}^2 \right] + \left[ \frac{1}{2\rho} \Pi_{r,-s}^2 + \frac{\rho\omega_r^2}{2} \Psi_{\kappa,-s,r}^2 \right] \right), \\
& = \mu_0^{N-1} \exp - \frac{i}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^\infty) \\
& \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{1,r}}{(2\pi)^{M_1}} \prod_{r=1}^{M_N} \frac{d\Pi_{-N,r}}{(2\pi)^{M_N}} \cdots \prod_{r=1}^{M_2} \frac{d\Pi_{-2,r}}{(2\pi)^{M_2}} \prod_{r=1}^{M_1} \frac{d\Pi_{-1,r}}{(2\pi)^{M_1}} \\
& \prod_{r=1}^{out+1} \exp - \frac{i}{\hbar} \left( \sum_{r=1}^{out+1} \left[ \frac{1}{4\rho} \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \Pi_{r,-s}\right) - \frac{1}{4\rho} \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \Pi_{r,s}\right) \sum_{s=1}^{N-1} \exp - \frac{i}{\hbar} \left( \sum_{r=1}^{out+1} \left[ \frac{\rho\omega_r^2}{2} \Psi_{\kappa,-s,r}^2 - \frac{\rho\omega_r^2}{2} \Psi_{\kappa,s,r}^2 \right] \right) \right], \right)
\end{aligned} \tag{231}$$

and we suppose that it is getting  $N$  is a prime number

$$\sum_{s=1}^{N-1} \exp - \frac{2\pi i \rho \omega_r^2}{4\pi \hbar} \left( \sum_{r=1}^{out+1} [\mathcal{N}_{\kappa,-s,r}^2 - \mathcal{N}_{\kappa,s,r}^2] \right) = \begin{cases} i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} - \sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}}, \\ -\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} + i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}}, \end{cases} \tag{232}$$

where  $\frac{4\pi\hbar}{\rho\omega_r^2}$  is congruent with 1 or 3 and modulus 4. The path integral is

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = \mu_0^{N-1} \exp - \frac{i(N+1)}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^\infty) \\
& \left( \int_{-\infty}^{\infty} \prod_{r=1}^{M_N} \frac{d\Pi_{r,s}}{(2\pi)^{M_N}} \prod_{r=1}^{M_N} \frac{d\Pi_{r,-s}}{(2\pi)^{M_N}} \exp - \frac{i}{\hbar} \sum_{r=1}^N \left( \sum_{r=1}^{out+1} \left[ \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \Pi_{r,-s}\right) - \frac{1}{4\rho} \frac{\hbar^2}{\mu_0^2} \cos\left(\frac{\sqrt{2}\mu_0}{\hbar} \Pi_{r,s}\right) \right] \right) \right)^N \\
& \begin{cases} (i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} - \sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}})^{N+1} \\ (-\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} + i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}})^{N+1}, \end{cases}
\end{aligned} \tag{233}$$

and path integral is

$$\begin{aligned}
& \langle \Pi_{out} t_{out} | \Pi_{in}, t_{in} \rangle = \mu_0^{N-1} \left( -\frac{2\hbar\rho\pi}{i} \right)^N \left( \frac{2\hbar\rho\pi}{i} \right)^N \exp - \frac{i(N+1)}{\hbar} ([\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r}]|_0^\infty) \\
& \begin{cases} (i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} - \sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}})^{N+1} \\ (-\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}} + i\sqrt{\frac{4\pi\hbar}{\rho\omega_r^2}})^{N+1}, \end{cases}
\end{aligned} \tag{234}$$

with this proceded it can be consider a complex integral.

## XI. PATH COMPLEX INTEGRAL

In this section, we try a complex extension that includes imaginary part; it is ignored this imaginary part. Here it is proposed a transformation that is complex in the path-integral.

In order to solve this integral

$$\mathcal{P}_{z,-n,r} = \exp(i \frac{\sqrt{2}\mu_0}{\hbar} \Pi_{-n,r}), \quad (235)$$

$$d\mathcal{P}_{z,-n,r} = i \frac{\sqrt{2}\mu_0}{\hbar} \mathcal{P}_{z,-n,r} d\Pi_{-n,r} \quad (236)$$

$$\mathcal{P}_{z,n,r-1} = \exp(i \frac{\sqrt{2}\mu_0}{\hbar} \Pi_{n,r-1}) \quad (237)$$

$$d\mathcal{P}_{z,n,r-1} = i \frac{\sqrt{2}\mu_0}{\hbar} \mathcal{P}_{z,n,r-1} d\Pi_{n,r-1} \quad (238)$$

the integral is

$$\begin{aligned} & \exp -\frac{i}{\hbar} \{ (\Psi_{n,\kappa,r} \Pi_n - \Psi_{-n,-\kappa,r} \Pi_{-n})(t_r - t_{r-1}) \} \\ & \left( \frac{\hbar^2}{2\mu_0^2} \int_0^{2\pi} \frac{d\mathcal{P}_{z,-n,r}}{(2\pi)^N} \mathcal{P}_{z,-n,r}^{-(\sqrt{2}\mu_0)^{-1} \Psi_{n,\kappa,r-1}} \right. \\ & \quad \left. \int_0^{2\pi} \frac{d\mathcal{P}_{z,n,r}}{(2\pi)^N} \mathcal{P}_{z,n,r}^{-(2\mu_0^{-1}) \Psi_{n,\kappa,r-1}} \right. \\ & \quad \exp i(t_r - t_{r-1}) \left( \frac{\hbar \mathcal{P}_{z,n,r}}{4\rho\mu_0^2} - \frac{\rho\omega_n^2}{2} \Psi_{\kappa,r-1,n}^2 \right) \\ & \quad \left( -\frac{\hbar^2}{2\mu_0^2} \int_0^{2\pi} \frac{d\mathcal{P}_{z,n,r-1}}{(2\pi)^N} \mathcal{P}_{z,n,r-1}^{-(2\mu_0)^{-1} \Psi_{n,\kappa,r-1}} \right) \\ & \quad \left. \int_0^{2\pi} \frac{d\mathcal{P}_{z,-n,r-1}}{(2\pi)^N} \mathcal{P}_{z,-n,r-1}^{-(\sqrt{2}\mu_0)^{-1} \Psi_{n,\kappa,r-1}} \right. \\ & \quad \exp i(t_r - t_{r-1}) \left( \frac{\hbar}{4\rho\mu_0^2} \mathcal{P}_{z,-n,r-1} \right. \\ & \quad \left. \left. - \frac{\rho\omega_n^2}{2} \Psi_{\kappa,r-1,n}^2 \right) \right) \quad (239) \end{aligned}$$

In this point it is important to mention that pathintegral is relized in polymeric quantum mechanics.

### A. Path Integral In One Dimension of a Crystal With Polymeric Quantum Mechanics

In one dimension structure is a toy model that is managed in a simple form, this to achieve the best understanding of structures that has space-time. In order to study complicated structure in space-time, and little scales, we abandon first extra- dimensions to analyze this type of structures.

With respect to very little scales, we consider Planck scales and path integral where it is always important to keep in mind that do not how is space-time that is governed by quantum mechanics.

The integral with dispersion is

$$\begin{aligned} & \langle \mathcal{P}_{z,r,OUT}, t_{OUT} | \mathcal{P}_{z,r,IN}, t_{IN} \rangle_{\text{Complex}\Pi} = \exp -\frac{i}{\hbar} \{ (\Psi_{\kappa,r} \Pi_r - \Psi_{-\kappa,-r} \Pi_{-r})(t_r - t_{r-1}) \} \\ & \frac{\hbar^2}{2\mu_0^2} \int_0^{2\pi} \frac{d\mathcal{P}_{z,r}}{(2\pi)^N} \mathcal{P}_{z,r}^{-(2\mu_0^{-1}) \Psi_{\kappa,r-1}} \left( \int_0^{2\pi} \frac{d\mathcal{P}_{z,-r}}{(2\pi)^N} \mathcal{P}_{z,-r}^{-(\sqrt{2}\mu_0)^{-1} \Psi_{n,\kappa,r-1}} \right) \\ & \exp \left[ -\frac{i}{\hbar} (t_r - t_{r-1}) \left( \frac{\rho_P^2}{\hbar} 2(l + \frac{1}{2})^2 (\Pi_{n,l}^2 c^2 + m_P^2 c^4) \Psi_{\kappa,r-1,-n}^2 \right) \right] \quad (240) \end{aligned}$$

where we have a term that is one temporal that is not an integral form, and it is equivalent in classical mechanics to a total temporal derivative between Hamiltonians that is a canonical transformation.

It is supposed a fragmented space-time in Planck scale and each piece oscillates in a big space that is indeterminate. But, we have a complex space and no real space. The imaginary part represents that part that is indeterminate that no has a contact.

In other hand, we establish with  $\Pi_{n,r} = \hbar\kappa_{n,r}$  here is important to mention that it is not the integral of this system. We have done a complex extension of these frequencies that they have form of momentum polymeric. Really we suppose that relationship for quantum is correct in a scale. We propose that there is a fractal structure; this idea is natural, and it is found for each known scale.

Now the integral with frequency and momentum is with  $l = -(r-1)$ .

where we suppose a pair of solutions that depend, if frequencies have relationship with momenta. Integral has two solutions that depends if frequency depends of momenta

$$\int \left( \frac{\mathcal{P}_{z,-(r-1)}^2}{(b\mathcal{P}_{z,-(r-1)}^{l=2,4})} \right) \Gamma(2, a\mathcal{P}_{z,-(r-1)}) d\mathcal{P}_{z,-(r-1)} =$$

$$\begin{cases} \frac{1}{b} (\mathcal{P}_{z,-(r-1)} \Gamma(2, a\mathcal{P}_{z,-(r-1)}) - \frac{(e^{(-a\mathcal{P}_{z,-(r-1)})} (a^2 \mathcal{P}_{z,-(r-1)}^2 + 2a\mathcal{P}_{z,-(r-1)} + 2))}{a}) + c_{const} & l=2 \\ \frac{1}{b} (Ei(-a\mathcal{P}_{z,-(r-1)}) - e^{(-a\mathcal{P}_{z,-(r-1)})} - e^{-a\mathcal{P}_{z,-(r-1)}} (a\mathcal{P}_{z,-(r-1)} + 1) \ln |a\mathcal{P}_{z,-(r-1)}| \\ + \ln |a\mathcal{P}_{z,-(r-1)}| \frac{1}{b} \Gamma(2, a\mathcal{P}_{z,-(r-1)})) + c_{const} & l=4 \end{cases} \quad (241)$$

where solution depends of frontiers of integration. It is impotant to mention that from 0 to  $\infty$  integral is limit of integrals.

## XII. CONCLUSIONS

Polymeric quantum mechanics is applied by means of a crystal in one dimension resulting  $\mu_0$  in terms of distances and disengage each oscillator. With this description, it is more convenient to use Euler product that Taylor development series of powers, and it was possible to describe better disengaging this system, where is considered usual classical mechanics. The polymeric quantum mechanics is managed, if we disengage the system first and thinking space-time as a crystal. In order to reduce this system. We think that this system in one dimension is similar to a space-time structure that has properties of a crystal that could be studied and understood with toy systems.

It is valid to think that exist "phonons" in this crystal. Usually, these phonons can be observed changing reference system. It is interesting to think that are these phonons or virtual particles found in the space-time in small scales. On the other hand, for some scales, it is gotten a structure that is found repeated in each scale that is already described in mathematics with fractal theories view for big structures as the universe. It is studied in no fundamental levels by means of complex canonical structure. The description is useful for each scale that is near to reality, and We obtain a higher-order theory that is reduced to a first-order theory [6, 18, 27]. It is important to mention that in this model fractal, is considering ordering of matter.

On the other hand, we calculate path integral for this model making with sums in position abandoning the continuous limit. In this point is making a extension to complex functions that in real part presents information of path integral,

## XIII. APPENDIX

We introduce mathematics used in this article. Appendix: Complex Analisis It is met that the complex integral in a closed path is zero, as complex Cauchy integral, but in our case, it is not closed path under real numbers, if we consider integral in theta

$$\int_0^{2\pi} d\theta \exp(iz) z' = \oint dz \frac{\exp iz}{(z - z_0)} = 2i\pi \exp(iz_0) \quad (242)$$

$$\begin{aligned} z &= R \exp i\theta = R \cos \theta + iR \sin \theta, \\ dz &= iR \exp i\theta = -R \sin \theta + iR \cos \theta, \end{aligned} \quad (243)$$

and the integral is

$$\begin{aligned} (iR) \int_0^{2\pi} d\theta \exp(iR\theta + iz) = \\ (iR) \int_0^{2\pi} d\theta \exp(-(R \sin \theta) \exp i(R\theta + R \cos \theta)) \end{aligned} \quad (244)$$

that says a relationship under this complex integral.

## A. Sum and Product.

This expression says a relationship between sum and product

$$\prod_{s=1}^{\infty} z_s = \prod_{s=1}^{\infty} R_s \exp(i\theta_s) = \exp\left(i\left(\sum_{s=1}^{\infty} \theta_s\right)\right) \prod_{s=1}^{\infty} R_s,$$

$$\sum_{s=1}^{\infty} \theta_s = -i \ln\left(\prod_{s=1}^{\infty} \frac{z_s}{R_s}\right), \quad (245)$$

that establish a relationship between sum and product in complex numbers.

### 1. Products and Sum of Complex quantity

In a general form for a complex number is written as

$$a_k = ||a_k|| \exp(ib_k) \quad (246)$$

and product of complex numbers

$$\prod_{k=1}^{\infty} \frac{a_k}{||a_k||} = \exp\left(i\sum_k b_k\right), \quad (247)$$

$$\sum_k b_k = -i \ln \left| \prod_{k=1}^{\infty} \frac{a_k}{||a_k||} \right|, \quad (248)$$

this says that is not a complex number, they are a lot complex number write as norm and angle.

## B. Sum of Sinus

We use a sinus and co-sinus sum or trigonometric relationship.

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta, \quad (249)$$

and a more complicated relationship, to demonstrate

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \frac{(\pi - 1)}{2}. \quad (250)$$

these permit to establish more complicated relationships.

## C. Euler's Products

There is a development of functions in products different to sum development

$$\frac{\sin\left(\pi \frac{rx}{(2n+1)a}\right)}{\pi \frac{r}{(2n+1)a} x} = \prod_{j=1}^{\infty} \left[1 - \left(\frac{rx}{j(2n+1)a}\right)^2\right] \quad (251)$$

$$\cos\left(\frac{\pi}{2} \frac{2rx}{(2n+1)a}\right) = \prod_{j=1}^{\infty} \left[1 - \left(\frac{2rx}{(2n+1)ja}\right)^2\right] \quad (252)$$

that establish products for these functions.

## D. Gamma Integral and Path Integral

Some quantities for  $\gamma$  and  $\Gamma$  that are used

$$\int dx (x^a e^{ibx}) = [-x^{(a+1)} (-ibx)^{-a-1} \Gamma(a+1, -ibx)], \quad (253)$$

$$\Gamma(a, x) = \int_x^\infty dt (e^{-t} t^{a-1}), \quad \gamma(a, x) = \int_0^x dt (t^{a-1} e^{-t}), \quad (254)$$

$$\Gamma(a, x) + \gamma(a, x) = \Gamma(a), \quad (255)$$

$$\Gamma(a, x) = e^{-x} (-1+a)! \sum_{k=0}^{a-1} \frac{x^k}{k!} \text{ for } a \text{ element } Z \text{ and } a > 0, \quad (256)$$

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \quad \gamma \text{ is Euler-Mascheroni constant} \quad (257)$$

where we must pay attention in gamma functions definitions. In another hand Gauss integral is

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad (258)$$

## E. Integral Gamma

$$\int \left(\frac{x^2}{(bx^{4,2})}\right) \Gamma(2, ax) dx = \begin{cases} \frac{1}{b} (Ei(-ax) - e^{(-ax)} - e^{-ax} (ax+1) \ln |ax| + \ln |ax| \frac{1}{b} \Gamma(2, ax)) + c_{const} n=4 \\ \frac{1}{b} (x \Gamma(2, ax) - \frac{(e^{(-ax)} (a^2 x^2 + 2ax + 2))}{a}) + c_{const} n=2 \end{cases} \quad (259)$$

## F. Path Integral

We consider an case where we get a non continuous medium

$$\sum_{n=0}^{p-1} e^{\frac{2\pi i}{p} n^2} = \begin{cases} \sqrt{p}, & p \equiv 1 \pmod{4} \\ i\sqrt{p}, & p \equiv 3 \pmod{4} \end{cases} \quad \sum e^{\frac{2\pi i r^2}{p}} \quad (260)$$

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## REFERENCES

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- [1] J. Smit; "Introduction to quantum Fields on a lattice: A robust mate", Cambridge Lect. Notes Phys. 15 (2002) 1 **271**.
  - [2] **R. Gambini, J. Olmedo, J. Pullin,**"Spherically symmetric loop quantum gravity: analysis of improved dynamics", (pdf), gr-qc, DOI: 10.1088/1361-6382/aba842, arXiv:2006.01513 [gr-qc]
  - [3] **P. Horava,** "Quantum Gravity at a Lifshitz Point," Phys. Rev. D **79** (2009) 084008 [arXiv:0901.3775 [hep-th]].

- [4] Morales Técotl, Hugo A.; Saeed, Rastgoo; Ruelas Juan C., Path Integral Polymer Propagator of Relativistic and Non-Relativistic Particle, (pdf), gr-qc, arXiv: 1608.04498v2.
- [5] Horava Petr, "Quantum Gravity at a Lifshitz Point", (pdf), hep-th arXiv:0901.3775v2.
- [6] Margalli C.A and Vergara J.D., *Hidden Gauge Symmetry in Holomorphic Models*, Phys.Lett. A379 (2015) 2434. From complex holomorphic systems to real systems.
- [7] Bailey, David H.; Swarztrauber, Paul N. (1994), "A fast method for the numerical evaluation of continuous Fourier and Laplace transforms" (PDF), SIAM Journal on Scientific Computing, 15 (5): 1105–1110, CiteSeerX 10.1.1.127.1534, doi:10.1137/0915067.
- [8] Boashash, B., ed. (2003), Time-Frequency Signal Analysis and Processing: A Comprehensive Reference, Oxford: Elsevier Science, ISBN 978-0-08-044335-5. Bochner, S.; Chandrasekharan, K. (1949), Fourier Transforms, Princeton University Press.
- [9] Bracewell, R. N. (2000), The Fourier Transform and Its Applications (3rd ed.), Boston: McGraw-Hill, ISBN 978-0-07-116043-8.
- [10] Gago R., Abrasonis G., Jiménez I. and Möller W, , Inc. (2007):"Growth mechanisms and structure of fullerene-like carbonbased thin films: superelastic materials for tribological applications", Chapter contribution for "Trends in Fullerene Research" To be published by Nova Science Publishers.
- [11] Nakayama Yu, (2014):Scale invariance vs conformal invariance, hep-th, arXiv:1302.0884v4.
- [12] Lapidus MichelL., American Mathematical Society, An Overview os Complex Fractal Dimension: From Fractal String to Fractal Drums, and Back, math-ph (2018), arXiv:1803.10399v2 [math-ph].
- [13] Calcagni G.,(2010):"Fractal universe and quantum gravity", Phys. Rev. Lett. 104, 251301 (2010), arXiv:0912.3142.
- [14] Salwinski D,ed.(2020), Euler's Sine Product Formula: An Elementary Proof, University of Toronto, The College Mathematics Journal · March 2018, DOI: 10.1080/07468342.2018.1419703.
- [15] A. Ashtekar, Lectures on Non-perturbative Canonical Gravity, World Scientific, Singapore 1991.
- [16] C. M. Bender, and S. Boettcher, Phys. Rev. Lett. 80, (1998) 5243.
- [17] Margalli C. A.,Mexico City UAM., Iztapalapa, J.David Vergara, Mexico U. ICN, May 26, 2020, From complex holomorphic systems to real systems, Published in: Int.J.Mod.Phys.A35 (2020) 13, 2050065
- [18] H. A. Morales-Tecotl, L. F. Urrutia, and J. D. Vergara, Reality conditions for Ashtekar variables as Dirac constraints, Class. Quant. Grav. 13, 2933 (1996) [gr-qc/9607044].
- [19] C.A. Margalli, J.D. Vergara, *Quantization of the Interacting Non-Hermitian Higher Order Derivative Field*, arXiv:1309.2928 [hep-th].
- [20] A. Ashtekar, Mathematical problems of nonperturbative quantum general relativity, (1992) [gr-qc/9302024].
- [21] Alejandro Corichi, Tatjana Vukašinac and José A. Zapata, Polymer Quantum Mechanics and its Continuum Limit, (2007)[gr-qc/0704.0007v2].
- [22] Ernesto Flores-González, Mexico City UAM., Iztapalapa, Hugo A. Morales-Técotl, Mexico, ESFM, Nov 14, 2019, Perturbative polymer quantum field theory and high energy scalar propagator, Class. Quant. Grav. 36 (2019) 24, 245007
- [23] Hugo A. Morales-Técotl, Mexico City U., Iztapalapa and Mexico, ESFM, Saeed Rastgoo,Mexico City U., Iztapalapa, Juan C. Ruelas, Mexico City U., Iztapalapa, 2018, Effective loop quantum geometry of Schwarzschild interior, e-Print: 1806.05795 [gr-qc].
- [24] Hugo A. Morales-Técotl(Mexico City U., Iztapalapa), Saeed Rastgoo(Mexico City UAM Iztapalapa, Juan C. Ruelas(Mexico City U., Iztapalapa), 2016, Path integral polymer propagator of relativistic and nonrelativistic particles Phys.Rev.D 95 (2017) 6, 065026 • e-Print: 1608.04498 [gr-qc].
- [25] Hugo A. Morales-Técotl, Mexico City U., Iztapalapa, Saeed Rastgoo,Mexico City U., Iztapalapa, Juan C. Ruelas,Mexico City U., Iztapalapa, 2016, Polymerization, the Problem of Access to the Saddle Point Approximation, and Thermodynamics, Phys.Rev.D 95 (2017) 6, 065026 , e-Print: 1608.04498 [gr-qc]. pdfDOI.
- [26] Hugo A. Morales-Técotl,Mexico City UAM, Iztapalapa, Daniel H. Orozco-Borunda,Mexico City UAM, Iztapalapa, Saeed Rastgoo, Mexico City UAM, Iztapalapa and Republica U., Montevideo , 2015, Polymer quantization and the saddle point approximation of partition functions, Phys.Rev.D 92 (2015) 10, 104029, e-Print: 1507.08651 [gr-qc].
- [27] A. Pais and G. E. Uhlenbeck, On Field theories with nonlocalized action, Phys. Rev. 79, 145 (1950).
- [28] Matej Pavsic,Pais-Uhlenbeck Oscillator and Negative Energies, (2016),e-Print: arxiv1607.06589 [gr-qc].
- [29] P. Horava, Quhttps://www.overleaf.com/project/5f21c7b8cf3c7b0001ca6dfcantum Gravity at a Lifshitz Point, (2009),Phys. Rev. D 79, , 084008[arXiv:0901.3775 [hep-th]].
- [30] Jasel Berra Montiel, Roberto Cartas, Deformation quantization and tomographic representation of quantum field, (2020), hep-th arXiv2006.07688v1.
- [31] Gorji M.A., Nozari K., Vakili B., Polymeric quantization and black hole thermodynamics, (2014), Physics Letters B, 735, 62-68.
- [32] Wei Gong, Mariano Mateos, John R. Singler, Yangwen Zhang, Analysis and approximations of Dirichlet boundary control of Stokes flows in the energy space, (2020), math Oc, arXiv:2011.08221v1.

1107 [33] . Denton Peter B, Parke Stephen J., Tao Terence, and Zhang Xining, Eigenvectors from Eigen Values: A  
1108 Survey of A Basic Identity In Linear Algebra, (2020), math.RA, arXiv:1908.03795v3.